Core Algebra: Lecture 4, Isomorphism Theorems[1]

Read Along:

Selick's notes: 1.1, 1.2.1, 1.4

Lang's book: I.1-3.

Recall from last time: For $H < G$, $G/H = \{[g]_H = \bar{g} = gH\}$.

$$g_1 \sim g_2 \Leftrightarrow [g_1] = [g_2] \Leftrightarrow g_2 = g_1 h,\ h \in H$$

If H is finite, $|[g]_H| = |H|$ so we get:

Theorem 2.1. *(**Lagrange's Theorem**) If G is finite, $|H| \mid |G|$ and*

$$|G| \,/\, |H| = |G/H| = (G : H) \text{ “the index of } H \text{ in } G\text{”}$$

If $N \triangleleft G$ $(N^g = N)$ then G/N is a group.

Theorem 2.2. *(**First Isomorphism Theorem**) If $\phi : G \to H$ is a morphism, then:*

$$G/ker\phi \cong im\phi$$

Also, if $\phi : G \to G/N = im\phi$ then $ker\phi = N$.

Goal:

Theorem 2.3. *(**Jordan-Hölder Theorem**) If G is finite, then we can write*

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \ldots \triangleright G_n = \{e\}$$

where G_i/G_{i+1} is “simple”, i.e. has no normal subgroups and any two such towers are “equivalent”.

Definition 2.4. For $K < G$, $N_G(K) := \{g \in G : \ K^g = K\}$.

Theorem 2.5. *(**Second Isomorphism Theorem**) If $H, K < G$, $H < N_G(K)$, then:*

1. *$N_G(K)$ is a group*
2. *$K < N_G(K)$*
3. *$N_G(K) = G \Leftrightarrow K$ is normal*

and also:

$$H \cap K \triangleleft H,\ K \triangleleft HK \text{(which is a group) and } HK/K \cong H/H \cap K$$

For a diagram presentation, see Figure 1.

[1] Notes from Professor Bar-Natan's Fall 2010 Algebra I class. All the mistakes are mine, please let me know if you find any! (ivahal@math.toronto.edu)

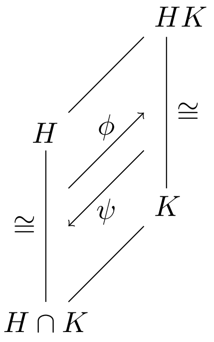

Fig. 1. The Second Isomorphism Theorem.

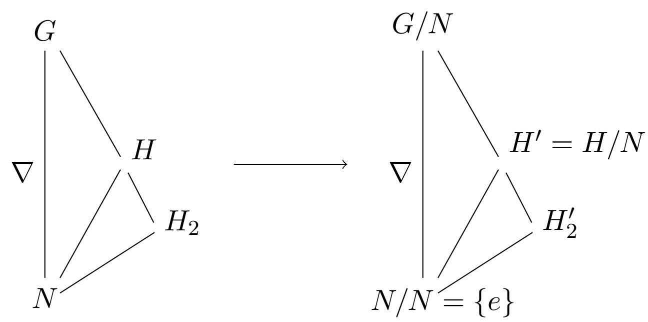

Fig. 2. The Fourth Isomorphism Theorem.

Proof. The steps are as follows:

1. $H \cap K$ is a group.
2. $H \cap K$ is normal in H.
3. HK is a group.
   Take any $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then $h_1k_1,\ h_2k_2 \in HK$ and:
   $$h_1k_1h_2k_2 = h_1h_2h_2^{-1}k_1h_2k_2 = h_1h_2k_1^{h_2}k_2 \in HK \text{ since } h_1h_2 \in H \text{ and } k^{h_2}k_2 \in K$$
4. $K \triangleleft HK$.
   Consider $k_1 \in K$ and $hk_2 \in HK$. Then, $k_1^{hk_2} = (k_1^h)^{k_2} \in K$.
5. Define $\phi([h]_{H\cap K}) = [h]_K$.
   a) Well-defined? For $t \in H \cap K \subset K$, $ht \mapsto [ht]_K = [h]_K$ so yes.
   b) Morphism? (easy)
6. Define $\psi([hk]_k) = [h]_{H\cap K}$.
   It is again easy to see this is well defined and a morphism.
7. ϕ, ψ are inverses of each other.

□

Theorem 2.6. *(***Third Isomorphism Theorem***) If $G \triangleright H$, $H > N$, $G \triangleright N$, then:*

$$G/N \triangleright H/N \text{ and } (G/N)/(H/N) \cong G/H$$

Proof. The first part is left as an exercise and for the second, define:

$$\begin{aligned} \phi : (G/N)/(H/N) &\rightarrow G/H \\ [[g]_N]_{H/N} &\mapsto [g]_H \\ \text{and } \psi : G/H &\rightarrow (G/N)/(H/N) \\ [g]_H &\mapsto [[g]_N]_{H/N} \end{aligned}$$

It is easy to see that these two maps are well-defined, morphisms and are inverses. □

Theorem 2.7. *(***Fourth Isomorphism Theorem***) If $N \triangleleft G$, then there is a bijection between subgroups of G that contain N and subgroups of G/N. This bijection preserves "subgroup", indices, intersections.*

For a diagram illustration see Figure 2.