The Plan:

$\{\text{field extensions}\} \xleftrightarrow[\text{Theorem}]{\text{Fundamental}} \{\text{groups}\}$

$\{\text{extensions by radicals}\} \longrightarrow \{\text{solvable groups}\}$

$S_{\mathbb{Q}}(3x^5 - 15x + 5) \longrightarrow \{S_5, \text{ not solvable}\}$

The 3 Isomorphism Theorems:

① Reminder: $\phi: R \to S$ a homomorphism
$\ker\phi \subset R$ is an ideal
$R/\ker\phi \cong \operatorname{im}\phi$

Similarly, in group theory:
$\phi: G_1 \to G_2$ is a group homomorphism
$\ker\phi \triangleleft G_1$
$\frac{G_1}{\ker\phi} \cong \operatorname{im}\phi$

② If $N \triangleleft H \triangleleft G$ and $N \triangleleft G$ (not necessarily true since not transitive)

$\Rightarrow \frac{G}{H} \cong \frac{G/N}{H/N}$

③ $H < G,\ N \triangleleft G \Rightarrow H/N\cap H \cong \frac{H\cdot N}{N}$

(partial) **Proof of: $H/N\cap H \cong \frac{H\cdot N}{N}$**

- $H/N\cap H = \{[h]_1 : h \in H\}$ where $h_1 \overset{1}{\sim} h_2$ if $h_1 = h_2 n$ where $n \in N \cap H$
- $\frac{H\cdot N}{N} = \{[h\cdot n]_2 : h \in H, n \in N\}$ where $h_1 n_1 \overset{2}{\sim} h_2 n_2$ if $h_1 n_1 = h_2 n_2 n$ for $n \in N$.
- Need to construct $\phi: \frac{H}{N\cap H} \to \frac{H\cdot N}{N}$ and $\psi: \frac{H\cdot N}{N} \to \frac{H}{N\cap H}$

1 is guaranteed to be in N

→ $\phi : [h]_1 \longmapsto [h \cdot 1]_2$

Claim: If $h_1 \overset{1}{\sim} h_2$, then $h_1 \overset{2}{\sim} h_2$. So ϕ is well-defined.
(always have to show this)

Proof: obvious. ■

→ $\psi : [hn]_2 \longmapsto [h]_1$

Claim: If $h_1 n_1 \overset{2}{\sim} h_2 n_2$ then $h_1 \overset{1}{\sim} h_2$

Proof: $h_1 n_1 \overset{2}{\sim} h_2 n_2 \Rightarrow$ for some $n \in N$, $h_1 n_1 = h_2 n_2 n$.
$\Rightarrow h_1 = h_2 n_2 n n_1^{-1}$
Need to show that $n_2 n n_1^{-1} \in N \cap H$:
- $n_2 n n_1^{-1} \in N$ since $n_2, n, n_1 \in N$
- $n_2 n n_1^{-1} = h_2^{-1} h_1 \in H$ ■

Definition: A group G is called "solvable" if $\exists$ a sequence of subgroups as follows:

$$G = H_K,\; H_K \triangleright H_{K-1} \triangleright H_{K-2} \triangleright \dots \triangleright H_1 \triangleright H_0 = \{0\}$$
s.t. for each i, H_i / H_{i-1} is Abelian.

Examples:

1. Every Abelian group is solvable.
$H_1 \trianglerighteq H_0$ where $H_1 = G$, $H_0 = \{e\}$

2. S_2 is solvable, as it is Abelian.

3. $S_3 = S(\Delta)$ is solvable. (3 rotations, no flips)

4. S_4

Proof of 3:

Aside: $\det A = \sum_\sigma (-1)^\sigma$?

$\sigma \rightarrow A_\sigma = \begin{pmatrix} \\ \end{pmatrix}$ $\quad \det A_\sigma = \pm 1$

3/4

MAT401

Mar. 28/08

$\phi : S_n \to \{\pm 1\}$ $\qquad$ σ is onto $\Rightarrow$ has a kernel

$\phi : \sigma \longmapsto \text{sign}(\sigma)$

Let $H_0 = \{e\}$, $H_1 = \ker\phi$, $H_2 = S_n$.

Then $\underset{\substack{\uparrow \\ \text{always} \\ \text{true}}}{H_0 \triangleleft H_1} \underset{\substack{\uparrow \\ \text{kernels} \\ \text{always} \\ \text{normal}}}{\triangleleft H_2}$ $\qquad$ $\frac{H_2}{H_1} \cong \{\pm 1\}$ by 1st Iso. Thm

$$H_1/H_0 = H_1 \underset{\substack{\uparrow \\ n=3}}{=} \underbrace{\left\{ \text{identity}, \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{matrix}, \begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{matrix} \right\}}_{\text{abelian}}$$

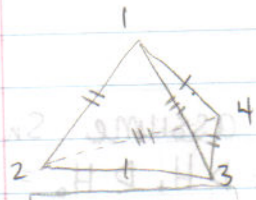
" C_3, $(\mathbb{Z}/3, +)$

Proof of 4: Need a theorem first.

Thm 1: If $N \triangleleft G$, then G is solvable $\Leftrightarrow$ G/N and N are solvable.

Proof of 4 continued:

$G = S_4 = S(\text{tetrahedron})$ is solvable



tetrahedron

has 3 pairs of edges where the elements in a pair don't intersect each other

σ is in S_4 (symmetry of T)

Let $\phi(\sigma)$ be the induced permutation of opposite pairs; $\phi(\sigma) \in S_3$

$\phi : S_4 \to S_3$

Example: $\phi\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

pair II $= \{\{2,1\},\{3,4\}\} \xleftarrow{\phi}$ pair I $= \{\{1,4\},\{2,3\}\}$ (same as $\{\{2,3\},\{4,1\}\}$)

pair I $= \{\{2,3\},\{4,1\}\} \xleftarrow{\phi}$ pair II $= \{\{1,2\},\{3,4\}\}$

pair III $= \{\{2,4\},\{3,1\}\} \xleftarrow{\phi}$ pair III $= \{\{1,3\},\{2,4\}\}$

$$\begin{array}{cccc} 11 & 10 & 01 & 00 \end{array}$$

Let $N = \ker\phi = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 \end{pmatrix} \right\}$

$G/N \cong S_3$ is solvable $\quad = \mathbb{Z}/2 \oplus \mathbb{Z}/2$

$|G| = |G/N||N| \quad (01)\circ(10) = 11$

$24 \quad 6 \quad 4 \quad$ Abelian $\Rightarrow$ solvable

Example 5: For $n \geq 5$, S_n is not solvable.

Proof: A_n = minimal subgrp of S_n containing all 3-cycles

$$= \begin{pmatrix} 1 & 2 & i & j & k & n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & j & k & i & n \end{pmatrix}$$

$$= (ijk)$$

Lemma 1: $[A_n, A_n] = A_n$

(Def. If $H < G$, then $[H,H]$ = minimal subgroup containing all $h_1h_2h_1^{-1}h_2^{-1} = [h_1, h_2]$ for $h_1, h_2 \in H$

* $[h_1, h_2]$ is "the commutator of h_1 & h_2". It is trivial if h_1 and h_2 are commutative.)

Lemma 2: If $H'' \rhd H'$ and $A < H''$ and H''/H' is Abelian then $[A, A] < H'$.

Lemma 1 & 2 prove example 5. Indeed, assume S_n was solvable. Then $S_n = H_k \rhd H_{k-1} \rhd H_{k-2} \rhd \cdots \rhd H_1 \rhd H_0 = \{e\}$ s.t. H_i/H_{i-1} is Abelian. $A_n < H_k \Rightarrow [A_n, A_n] < H_{k-1}$ but $A_n = [A_n, A_n]$, so $A_n < H_{k-1}$. It follows that $A_n < H_{k-2}$.

might not in book

By induction, $A_n < H_{k-2} \Rightarrow \cdots \Rightarrow A_n < H_0$, but $A_n \neq \{e\}$ so $\Rightarrow\Leftarrow$

Proof of Lemma 2:

Let $a, b \in A$. Consider $aba^{-1}b^{-1} \in [A, A]$. In H''/H', $[aba^{-1}b^{-1}]$ $= \underbrace{[a][b][a]^{-1}[b]^{-1}}_{\text{in an Abelian grp, by assumption}} = [e]$. So $aba^{-1}b^{-1} \in H'$. So H' contains the minimal subgroup containing all commutators, i.e. $H' \supset [A, A]$.

Proof of Lemma 1:

Consider $[(ijk)(klm)]$ for $i \neq j \neq k \neq l \neq m$ $(n \geq 5)$.

$$[(ijk)(klm)] = (ijk)(klm)(ijk)^{-1}(klm)^{-1}$$
$$= (ijk)(klm)(jik)(mlk)$$
$$= \begin{pmatrix} 1 & \cdots & i & j & k & l & m & n \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ & & l & j & i & k & m & \end{pmatrix}$$

not touched

$$= (ilk) \Rightarrow [A, A] \text{ contains all 3-cycles.} \blacksquare$$

Proof of Thm 1 (If $N \triangleleft G$ then N & G/N solvable $\Leftrightarrow$ G is solvable):

$\Rightarrow$: Assume N & G/N are solvable.

$N = H_k \triangleright H_{k-1} \triangleright \ldots \triangleright H_1 \triangleright H_0 = \{e\}$

$G/N = M_l \triangleright M_{l-1} \triangleright \ldots M_1 \triangleright M_0 = \{N\}$ $(N = [e])$

s.t. H_i/H_{i-1} and M_j/M_{j-1} are Abelian.

Consider $\pi : G \to G/N$, $g \xrightarrow{\pi} [g]$ (onto)

Consider $\underbrace{\{e\} = H_0 \triangleleft \ldots \triangleleft H_K = N}_{\text{nothing to prove}} = \underbrace{\pi^{-1}(M_0) < \pi^{-1}(M_1) < \pi^{-1}(M_2) \ldots < \pi^{-1}(M_l) = G}_{\text{need to treat}}$

Claim: $\pi^{-1}(M_{j-1})$ is normal in $\pi^{-1}(M_j)$ and $\dfrac{\pi^{-1}(M_j)}{\pi^{-1}(M_{j-1})}$ is Abelian.

Reminder: $N \triangleleft G \Rightarrow n \in N, g \in G, g^{-1}ng \in N$

Assume $a \in \pi^{-1}(M_{j-1})$, $b \in \pi^{-1}(M_j)$

$a \in \pi^{-1}(M_{j-1}) \equiv [a] \in M_{j-1}$.
$b \in \pi^{-1}(M_j) \equiv [b] \in M_j$

$\pi(b^{-1}ab) = [b^{-1}ab] = [b]^{-1}[a][b]$
$[b]^{-1}[a][b] \in M_{j-1}$, as $M_{j-1} \triangleleft M_j$
$b^{-1}ab \in \pi^{-1}(M_{j-1})$. ∎

$\pi|_{\pi^{-1}(M_j)} = \pi_j$ (restricted)

$\pi_j : \pi^{-1}(M) \to G/N$
$\operatorname{im} \pi_j = \pi(\pi^{-1}(M_j)) = M_j \subset G/N$
$\ker \pi_j = N$
$\Rightarrow$ By 1st Isomorphism Thm, $\frac{\pi^{-1}(M_j)}{N} \cong M_j$

$$\frac{M_j}{M_{j-1}} \cong \frac{\pi^{-1}(M_j)/N}{\pi^{-1}(M_{j-1})/N}$$

$$\cong \frac{\pi^{-1}(M_j)}{\pi^{-1}(M_{j-1})} \quad \text{by 2nd Iso. Thm} \ ∎$$

$\frac{1}{2} \Leftarrow$: $N \triangleleft G$, G solvable $\Rightarrow G/N$ is solvable.

<u>Proof</u>: $G = H_k \triangleright \dots \triangleright H_j \triangleright \dots \triangleright H_0 = \{e\}$, H_k/H_{k-1} Abelian
$H_j \to \frac{H_j \cdot N}{N} > \frac{H_{j-1} \cdot N}{N} \dots\dots = G/N$

<u>Claim</u>: $\frac{H_{j-1}N}{N}$ is normal in $\frac{H_jN}{N}$ and their quotient is Abelian.

<u>Proof</u>: Take $a \in \frac{H_{j-1}N}{N}$ and $b \in \frac{H_jN}{N}$

$a = [h_a n_a]$, $b = [h_b n_b]$
$b^{-1}ab = [\underbrace{h_b^{-1} h_a h_b}]$ = something in equiv. class of h_{j-1}
$h_b \in H_j$, $h_a \in H_{j-1} \to \in H_{j-1}$

MAT401
Mar 28/08

$$\frac{H_j \cdot N/N}{H_{j-1}N/N} = \frac{H_jN}{H_{j-1}N}$$ (2nd Iso. Thm)

Let $A = \dfrac{H_j}{H_{j-1}}$, $B = \dfrac{H_jN}{H_{j-1}N}$

$A \xrightarrow{\phi} B$

ϕ is onto

$B = A/\ker\phi$, so B is Abelian.

$$\phi([h]_A) = [h]_B$$

$h \in H_j$

Claim: 1. ϕ is well-defined

2. ϕ is onto.