MAT240 LECTURE 9

Oct 6/14

READ ALONG: SEC 1.5-1.7

EXAMPLES: In $V = M_{2\times2}(\mathbb{F})$

$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\}\ S_1$

$N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $N_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $N_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $N_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $\}\ S_2$

Claims

- $M_1 \ldots M_4$ generate (or span) V, indeed

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underline{a}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \underline{b}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \underline{c}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- $N_1 \ldots N_4$ generate V

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underline{\quad}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \underline{\quad}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \ldots$$

Aside: 1. If W is a subspace of V and S is contained in W ($S \subseteq W$), then $\text{Span}(S) \subseteq W$

Proof b/c W is closed under addition and scalar multiplication, hence under linear combinations.

Aside: 2. If $S_2 \subseteq \text{Span}(S_1)$ then $\text{Span}(S_2) \subseteq \text{Span}(S_1)$

Proof:

Use aside #1 w/ $W = \text{span}(S_1)$

re elements of S_2 lin. comb. of elements in S_1

"linear combination of linear combinations are linear combinations"

Subexample (i.e. re: the "Asides")

$u_1, u_2 \in S_1$ $\qquad S_2 = \left\{ \begin{matrix} 7u_1 + 3u_2 \\ 2u_1 - u_2 \end{matrix} \right\} \begin{matrix} \to v_1 \\ \to v_2 \end{matrix}$

$u \in \text{span}(S_2)$ e.g. $u = 2v_1 - v_2$

$$u = \underbrace{2(7u_1 + 3u_2) - (2u_1 - u_2)}_{\text{l.c. of l.c.}} = \underbrace{12u_1 + 7u_2}_{\text{is a l.c.}}$$

Aside 3

If S_1 spans V & $S_1 \subseteq \text{span}(S_2)$ then S_2 spans V.

Indeed, by Aside 2, $V = \text{span}(S_1) \subseteq \text{span}(S_2) \subseteq V$

$\Rightarrow \text{Span}(S_2) = V \Rightarrow S_2$ spans V

---

back to example

To show that $\text{span}(S_2) = M_{2\times 2}(F_5)$, by Aside 3 it is enough to show that $M_1 \in \text{span}(S_2)$, $M_2 \in \text{span}(S_2)$ etc.

Indeed $M_1 = \frac{1}{3}\left(N_1 + N_2 + N_3 + N_4 - 3N_1\right) = \underbrace{-\frac{2}{3}N_1 + \frac{1}{3}N_2 + \frac{1}{3}N_3 + \frac{1}{3}N_4}_{\in \text{span}(S_2)}$

$$\frac{1}{3}\left(\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} - 3\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right) =$$

can similarly we can do same for M_2, M_3, M_4.