Theorems of Chapter 15:

- **Definition:** Suppose ϕ R $\rightarrow$ S, where R and S are rings

If: $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\,\phi(b)$ $\quad \forall\, a, b \in R$
Then: ϕ is a Ring *Homomorphism*

In addition, if ϕ is 1:1 and onto, then ϕ is a Ring *Isomorphism*

- $k \rightarrow k \bmod n$ is called the *natural ring homomorphism* that takes $Z \rightarrow Z_n$

- **Theorem 15.2 Kernals are Ideals**
  Let ϕ: R $\rightarrow$ S be a ring homomorphism
  Then Ker $\phi = \{ r \in R : \phi(r) = 0 \}$ is an ideal of R

- **Theorem 15.3 First Isomorphism Theorem for Rings**
  Let ϕ: R $\rightarrow$ S be a ring homomorphism
  Then R/Ker $\phi \approx \phi(R)$

- **Theorem 15.4 Ideals are Kernals**
  Suppose I = ideal of R
  Then I = Kernal of a ring homomorphism
  In particular, Ideal A is the kernel of ϕ: $R \rightarrow R/A$

- **Theorem 15.5 Homomorphism from Z to a ring with unity**
  Let R be a ring with unity
  Then ϕ: Z $\rightarrow$ R, n $\rightarrow$ n.1 *is a ring homomorphism*

- **Corollary 1 A ring with unity contains Z_n or Z**
  Suppose R is a ring with unity, if:
  1. char R = n > 0 $\Rightarrow$ R has a subring isomorphic to Z_n
  2. char R = 0 $\Rightarrow$ R has a subring isomorphic to Z

- **Corollary 2 Z_m is a homomorphic image of Z**
  Suppose ϕ: Z $\rightarrow$ Z_m given by x $\rightarrow$ x mod m $\forall$ positive integer m
  Then ϕ is a ring homomorphism

- **Corollary 3 A field contains Zp or Q**
  Suppose F is a field, if:
  1. char F = p $\Rightarrow$ F has a subfield isomorphic to Z_p
  2. char F = 0 $\Rightarrow$ F has a subfield isomorphic to Q

- **Theorem 15.6 Field of Quotients**
  Let D = integral domain
  Then $\exists$ *field of quotient of D* (F), <u>such that</u> F contains a subring isomorphic to D