1. The Simplicity of the Alternating Groups

This handout is to be read twice: first read red only, to ascertain that everything in red is easy and boring, then read black and red , to actually understand the proof.

Theorem 1.1. *The alternating group $A_n \trianglelefteq S_n$ is simple for $n \neq 4$.*

Remark 1.2. Easy for $n \leq 3$, false for $n = 4$ as there is $\phi : A_4 \twoheadrightarrow A_3$, so assume $n \geq 5$.

Lemma 1.3. *Every element of A_n is a product of 3-cycles.*

Proof: Every $\sigma \in A_n$ is a product of an even number of 2-cycles, and (12)(23)=(123) and (123)(234)=(12)(34).

Lemma 1.4. *If $N \trianglelefteq A_n$ contains a 3-cycle, then $N = A_n$.*

Proof: WLOG, $(123) \in N$. Then for all $\sigma \in S_n$, $(123)^\sigma \in N$:

If $\sigma \in A_n$, this is clear. otherwise $\sigma = (12)\sigma'$ with $\sigma \in A_n$, and then as $(123)^{(12)} = (123)^2$, $(123)^\sigma = ((123)^2)^{\sigma'} \in N$. So N contains all 3-cycles.

Case 1. N contains an element with cycle of length ≥ 4.
Resolution. $\sigma = (123456)^{\sigma'} \in N \Rightarrow \sigma^{-1}(123)\sigma(123)^{-1} = (136) \in N$.

Case 2. N contains an element with 2 cycles of length 3.
Resolution. $\sigma = (123)(456)^{\sigma'} \in N \Rightarrow \sigma^{-1}(124)\sigma(124)^{-1} = (14263) \in N$.

Case 3. N contains $\sigma = (123)$(a product of disjoint 2-cycles).

Resolution. $\sigma^2 = (132) \in N$.

Case 3. Every element of N is a product of disjoint 2-cycles.
Resolution. $\sigma = (12)(34)^{\sigma'} \Rightarrow \sigma^{-1}(123)\sigma(123)^{-1} = (13)(24) = \tau \in \in N$ $\Rightarrow \tau^{-1}(125)\tau(125)^{-1} = (13452) \in N$.