

Last class had many details \rightarrow don't memorize them!
 There was also structure \rightarrow know this
 Look at "Unit Summary" online.
 You should be able to reproduce all the details of the proofs & have an image of the proofs & meanings in your mind.

Claim: Every linearly independent set L in a finite-dimensional (has finite basis) vector space V can be extended to a basis of V .

Pf:

1. Add elements to L , one by one, making sure that each additional element is not in the span of what you had before. The result remains linearly independent.
2. L , take G to be some basis of V , use replacement.

Get $H \subset G$ s.t. $|H| = |G| - |L|$ & $H \cup L$ generates and is an extension of L .

$$|H \cup L| \leq |H| + |L| = |G| - |L| + |L| = |G| = n = \dim V.$$

but $|H \cup L| \geq n \Rightarrow |H \cup L| = n$
 So $H \cup L$ is a basis. \square

Thm: Assume V is finite-dimensional and $W \subset V$ is a subspace.

1. W is also finite-dimensional (f.d)
2. $\dim W \leq \dim V$
3. Every basis of W can be extended to be a basis of V .

Pf: 1. Pick elements of W one by one, making sure that no one is in the span of the previous ones

Choose $w_1 \in W$ s.t. $w_1 \neq 0$

(if possible, $W = \{0\}$, \emptyset is a basis)

Choose $w_2 \notin \text{span}(w_1)$ (if impossible w_1 is a basis)
 $\in W$



Assume w_1, \dots, w_k were chosen in W s.t. (w_1, \dots, w_k) is linearly independent. If it generates, it is a basis and we're done. If not, there exists $w_{k+1} \in W$, $w_{k+1} \notin \text{span}(w_1, \dots, w_k)$. So w_1, \dots, w_{k+1} is still lin. indep. and we can continue. The process is guaranteed to stop for some $k \leq n$, as every lin. indep. set in V has at most n elements.

2. Any basis α of W is lin. indep. in V so $|\alpha| \leq \dim V$
 $\dim W$

3. If $\alpha \subset W$ is a basis, it is lin. indep., hence by previous claim it can be extended.

hm. Doc thinks it's fishy to use axiom of choice.

Every vector space has basis.
 $P(X) = \{1, x, x^2, \dots\} \rightarrow \infty$ basis

Pf. uses "the axiom of choice"

Eg. $F = \mathbb{Q}$. (rational #'s)
 $V = \mathbb{R}$. (real #'s)

Search for a basis.

Take it from \mathbb{R} .

$1, \sqrt{2}, \sqrt[3]{7}, \dots, \pi, e$

$$\frac{a}{b} + \frac{c}{d}\sqrt{2} + \frac{e}{f}\sqrt[3]{7}$$

Open problem: is $\pi + e$ rational?

People have proven that π and e are rational, but Moore has been able to prove that $\pi + e$ is rational.

So we don't know whether we need e in our basis or not. So by claiming that \mathbb{R} has basis is claiming you've proven the $\pi + e$ problem.