

MAT240: Abstract Linear Algebra Lecture:

Determinants:

1. Applications: Mention two, prove one, use none.
2. Formulas: discuss just one.
3. Basic Properties. Our core subject.

Det: $M_{n \times n}(F) \rightarrow F$

Use 1:

$$\det \begin{pmatrix} - & - & - & r_1 & - & - & - \\ & & & \vdots & & & \\ - & - & - & r_n & - & - & - \end{pmatrix} = \text{vol}(\text{parallelepiped spanned by } r_1 \dots r_n)$$

Use 2:

$$\det(A) \neq 0 \quad \text{iff} \quad A^{-1} \exists$$

Definition:

$$|(a_{11})| := a_{11}$$

$$|A| = \left| \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \right| \equiv \sum_{j=1}^n (-1)^{i+j} a_{ij} |\widetilde{A}_{ij}|$$

in general, \widetilde{A}_{ij} is the matrix obtained from A by removing the i^{th} row and j^{th} column.

$$\text{Eg. } \left| \begin{pmatrix} a & c \\ b & d \end{pmatrix} \right| = ad - bc$$

$$\text{Eg. } \left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right| = (-1)^{1+1} * a_{11} |(a_{22})| = a_{11} a_{22} - a_{12} a_{21}$$

$$\text{Eg. } \left| \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right| = 1 * 4 - 2 * 3 = -2$$

$$\text{Eg. } \left| \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \right| = 1 \left| \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} \right| - 2 \left| \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} \right| + 3 \left| \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} \right| = 1(-3) - 2(-6) +$$

$$3(-3) = 0$$

→ The above matrix is not invertible $\therefore \text{rank} < 3$

$$\text{Eg. } \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0 - 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1(-1) + 1(1) = 2 \rightarrow \text{this matrix is}$$

invertible.

Theorem: The following basic properties hold:

0. $\det(I_n) = 1$

1. $\det(E_{ij}^1 A) = -\det(A) \rightarrow \det(E_{ij}^1) = -1$

“flipping two rows in A, flips sign of det A.”

Eg. $\det \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = 3 * 2 - 4 * 1 = 2$

2. $\det(E_{1c}^2 A) = c * \det(A) \rightarrow \det(E_{1c}^2) = c$, even if $c = 0$

“multiplying a row by c multiplies det A by c.”

Eg. $\det \begin{pmatrix} 1 & 2 \\ 21 & 28 \end{pmatrix} = 28 - 42 = -14 = 7(-2)$

Eg. $\det \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = 0 * 4 - 0 * 3 = 0$

Moral: det of a matrix with a row of 0's is 0.

3. $\det(E_{1jc}^3 A) = \det A \rightarrow \det(E_{1jc}^3) = 1$

“Adding a multiple of one row to another row to another does not change det A”.

E.g. $\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} \xrightarrow{r_2 += 2r_1} \begin{vmatrix} 1 & -2 \\ 5 & 8 \end{vmatrix} = -2$

Claim: Using this we can compute $\det(A)$ for any A.

Method: Row reduce A to B which is RREF, keeping track of the changes to det.

If B is RREF:

$$\det B = \begin{cases} 1 \text{ if } B = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix} \\ 0 \text{ if } B = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \\ 0 & \dots & 0 \end{pmatrix} \end{cases}$$