MAT240 Oct. 26, 2006.

Plan 1. All you can say about linear transformations w/o fixing a basis.

2. Then choose a basis.

Recall: $T: V \to W$ is a "lin. trans" if

1. $T(0) = 0$
2. $T(x+y) = T(x) + T(y)$ claim $\Leftrightarrow$ $T(cx+y) = cT(x) + T(y)$ $x, y \in W$ $c \in F$.
3. $T(cx) = cT(x)$ $x \in V$

Examples. 1. $0: V \to W$ $0(x) = 0_W$

2. (1) $Id_V = I_V$ "The identity trans of V"

$I_V: V \to V$ $I_V(x) = x$

$I_V(cx+y) \overset{?}{=} cI_V(x) + I_V(y)$

$cx+y \overset{!}{=} cx+y$.

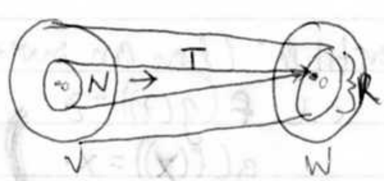
Given any linear transformation $T: \overset{N(T)}{V} \to \overset{R(T)}{W}$,

Def'n 1. $N(T) = \ker T$ = Null space of T / Kernel of $T = \{x \in V \mid T(x) = 0_W\}$

$N(0) = V$

$N(Id_V) = \{0_V\}$

2. $R(T) = \mathrm{Im}(T)$ = Range of T / Image of T $= \{w \in W \mid \exists x \in V, w = Tx\}$
$= \{Tx \mid x \in V\}$



$R(0) = \{0_W\}$

$R(Id_V) = V$

Proposition 1. $N(T)$ is a subspace of V
2. $R(T)$ is a subspace of W

(½) **Pf (1)** 1. $x, y \in N(T)$ $\quad (x+y) \overset{?}{\in} N(T)$

$\Rightarrow T(x) = 0 = T(y)$

$\Rightarrow T(x+y) = T(x) + T(y) = 0 + 0 = 0$

$\Rightarrow (x+y) \in N(T)$

Pf(2) 2. $w_1, w_2 \in R(T)$

$\Rightarrow \exists\, x_1 \in V,\ x_2 \in V$ s.t. $w_i = T(x_i)$

So $T(x_1 + x_2) = T(x_1) + T(x_2) = w_1 + w_2$

Hence $w_1 + w_2 \in R(T)$

Def'n 1. $\dim N(T) =: \text{nullity}(T)$
2. $\dim R(T) =: \text{rank}(T)$

Proposition: (Aside: $f : A \to B$

1. f is "1-1" if $f(x) = f(y) \Rightarrow x = y$
2. f is "onto" if $\forall z \in B\ \exists x \in A$ s.t. $F(x) = z$.

f is 1-1 and onto $\Leftrightarrow$ f is "invertible" (has an inverse)
$\exists g : B \to A$ s.t. $f(g(z)) = z$
$g(f(x)) = x$)

1. $\text{nullity}(T) = 0 \Leftrightarrow T$ is 1-1 $\quad N(T) = \{0\}$
2. $\text{rank}(T) = \dim W \Leftrightarrow T$ is onto.

MAT240

Pf 2. $R(T) \subset W$

rank $= \dim R(T) = \dim(W) \Leftrightarrow R(T) = W$

$\Leftrightarrow$ T is onto

Pf 1. ($\Leftarrow$) Assume T is 1-1.

Let $x \in N(T)$, then $T(x) = 0 = T(0)$

So $T(x) = T(0)$, by 1-1 $\Rightarrow x = 0$

$\Rightarrow N(T) = \{0\} \Rightarrow$ nullity $(T) = 0$.

($\Rightarrow$) Assume nullity $(T) = 0 \Rightarrow N(T) = \{0\}$.

Assume $T(x) = T(y) \Rightarrow T(x) - T(y) = 0$

$\Rightarrow T(x-y) = 0$

$\Rightarrow x - y \in N(T)$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$ $\square$

use identity & 0 maps as examples

Thm. (The dimension theorem / rank - nullity thm)

Let V be a finite-dimensional v.s. Then,

rank (T) + nullity $(T) = \dim V$

Example $D_{-1} : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ $\quad D_{-1}(P) = P'$

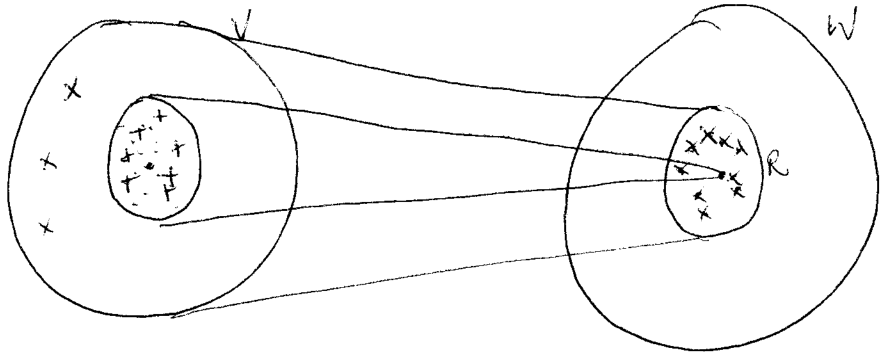
$D_0 : \underbrace{P_n(\mathbb{R})}_{V} \to P_n(\mathbb{R})$ $\quad D_0(P) = P'$

	T	N(T)	nullity(T)	R(T)	rank(T)	dim V
onto	D_{-1}	{constants}	1	$P_{n-1}(\mathbb{R})$	n	n+1
not onto	D_0	{constants}	1	$P_{n-1}(\mathbb{R})$	n	n+1

everything

not everything

Pf.



$N(T) \subset V$ So $N(T)$ is finite-dimensional.
Pick a basis $\{v_1 \dots v_k\}$ of $N(T)$. By an earlier theorem, you can find $\{v_1 \dots v_\ell\} \in V$ s.t. $\{u_1 \dots u_k, v_1 \dots v_\ell\}$ is a basis of V.

Claim: Let $w_i = T(v_i)$. Then $\{w_i\}$ is a basis of $R(T)$
If so,

$$\dim V = k + \ell$$
$$\dim N(T) = k$$
$$\dim R(T) = \ell$$ and rank-nullity thm is proven.

Pf of claim $\{w_i\}$ span

$$R(T) = \{Tv : v \in V\} = \{T(\sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{\ell} b_i v_i)\} =$$

$$= \{\sum a_i \underbrace{T(u_i)}_{0} + \sum b_i T(v_i)\}$$

$$= \{\sum b_i w_i\} = \text{span}(w_1 \dots w_\ell)$$

Now need to show
$\{w_i\}$ is lin indep. Assume $\sum b_i w_i = 0$ Then
$T(\sum b_i v_i) = \sum b_i T(v_i) = \sum b_i w_i = 0$
$\Rightarrow \underbrace{\sum b_i v_i}_{G} \in N(T)$ therefore, $\exists a_i$ s.t.
$G = \sum a_i u_i$

$\sum b_i v_i = G = \sum a_i u_i$
$\Rightarrow \sum a_i u_i + \sum (-b_i) v_i = 0$
but $\{u_i, v_i\}$ is a basis,
$\Rightarrow$ $a_i = 0 \ \forall i$, $b_i = 0 \ \forall i$ $\Rightarrow b_i = 0 \ \forall i$ $\square$

Thm: $\text{nullity}(T) + \text{rank}(T) = \dim V$

MAT240

<u>Corrolary:</u> TFAE; : (The following are equivalent)
when $\dim V = \dim W$ & $T: V \to W$

$\Updownarrow$ 1. T is 1-1
$\Updownarrow$ 2. T is onto
$\Updownarrow$ 3. $\text{rank } T = \dim V$
$\Updownarrow$ 4. $\text{nullity } T = 0$
5. T is invertible

$1 \Leftrightarrow 4$ ✓
$2 \Leftrightarrow 3$ ✓
$1 \Rightarrow 2$ ✓ rank + nullity = dim
rank + 0 = dim
rank = dim
$\Rightarrow 3 \Rightarrow 2$ ✓
$2 \Rightarrow 1$ ✓ above backwards
$5 \Leftrightarrow 1, 2$ ✓

<u>Thm.</u> (challenge) Assume $T: V \to W$, $T': V' \to W'$ s.t.
$(\dim V, \dim W, \text{rank } T) = (\dim V', \dim W', ~~\dim~~ \text{rank } T')$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\Phi"'] & W \arrow[d, "\Psi"] \\ V' \arrow[r, "T'"] & W' \end{tikzcd}$$

Then $\exists$ an isomorphism $\Phi: V \to V'$
$\Psi: W \to W'$ s.t.
$\forall v \in V \quad \Psi(T(v)) = T'(\Phi(v))$