UNIVERSITY OF TORONTO

Faculty of Arts and Sciences

FINAL EXAMINATIONS, APRIL-MAY 2008

Math 401H1S Polynomial Equations and Fields

Instructor: Dror Bar-Natan
Date: April 28, 2008

Duration. You have 3 hours to write this exam.

Allowed Material. Basic calculators, not capable of displaying text or sounding speech.

Solve 6 of the following 7 questions. Each question is worth 17 points, to a maximum possible total of 102. Different parts of the same question may be weighted differently. You will get 4 points total for any problem for which you will write explicitly “I don’t know how to solve this problem” (whole problems only!).

Neatness counts! Language counts! The *ideal* written solution to a problem looks like a proof from the textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like “there exists” or “for every” cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.

Good Luck!

Solve 6 of the following 7 problems. Neatness counts! Language counts!

Problem 1. Let R be a commutative ring with unity and let A be an ideal of R. Define "A is prime" and "a ring D is a domain" and prove that $D := R/A$ is a domain if and only if A is prime.

Tip. The phrase "if and only if" means that there are two things to prove.

Problem 2. Let $\mathbb{Q}$ be the ring of rational numbers and let $\mathbb{Z}$ be the ring of integers.

1. Is there a ring S and a ring homomorphism $\phi : \mathbb{Q} \to S$ so that $\ker \phi = \mathbb{Z}$?
2. Is there a ring S and a ring homomorphism $\psi : \mathbb{Q} \to S$ so that $\operatorname{im} \psi$ is isomorphic to $\mathbb{Z}$?

Tip. These, of course, are not just yes/no questions. You are expected to fully justify your answers, whatever they are.

Problem 3. Let F be a field, A a non-zero ideal in $F[x]$, and $g \in F[x]$ a polynomial. Prove that $A = \langle g \rangle$ if and only if g is a non-zero polynomial of minimal degree in A.

Tip. As always in math exams, when proving a theorem you may freely assume anything that preceded it but you may not assume anything that followed it.

Problem 4. Is it always true that a splitting extension of a splitting extension is a splitting extension? In other words, let F be a field, $f \in F[x]$ be a polynomial with coefficients in F, K be a splitting field of f over F, $g \in K[x]$ be a polynomial with coefficients in K, and E be a splitting field of g over K. Is it always the case that there is a polynomial $h \in F[x]$ with coefficients in F so that E is a splitting field of h over F?

Tip. This, of course, is also not just a yes/no question. Whatever you state, you have to prove, unless it is a known earlier result.

Problem 5. Let E/F be a field extension, and let a_k (for $0 \leq k \leq n$) be elements of E that are algebraic over F. Let b be some solution in E of the equation $\sum_{k=0}^{n} a_k b^k = 0$. Prove that b is also algebraic over F.

Problem 6. For any group A, recall that $[A, A]$, the commutator group of A, is the subgroup of A generated by all elements of the form $[x, y] := xyx^{-1}y^{-1}$, where $x, y \in A$.

1. Let G be a group. Define "G is solvable".
2. For any group H, prove that if $H' \lhd H$ is a normal subgroup, if H/H' is Abelian and if $A < H$ is some other subgroup, then $[A, A] < H'$.
3. Prove that if a group G contains a non-trivial subgroup A for which $[A, A] = A$, then G is not solvable.

Problem 7. Let F be the field $\mathbb{Q}(i)$ (note that F is not $\mathbb{Q}$!) and let E be the field $\mathbb{Q}(\sqrt[4]{2}, i)$.

1. Compute $G := \mathrm{Gal}(E/F)$.
2. Find all the subgroups H of G.
3. For exactly one non-trivial proper subgroup of G (that is, a subgroup that is neither $\{e\}$ nor G), describe the fixed field E_H.

Tip. The word "describe" here means "find $a \in E$ so that $E_H = F(a)$".

Good Luck!