Examples

1. $L: R^2 \to R^3$ by $L\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a-b \\ 0 \\ a+7b \end{pmatrix}$

Linear

Not Linear

2. $L\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a^2 \\ |b| \\ a \end{pmatrix}$

$L(cx+y) = L\begin{pmatrix} ca+a' \\ cb+b' \end{pmatrix} = \begin{pmatrix} (ca+a')^2 \\ |cb+b'| \\ ca+a' \end{pmatrix}$

$x = \begin{pmatrix} a \\ b \end{pmatrix} \quad y = \begin{pmatrix} a' \\ b' \end{pmatrix}$

$\stackrel{?}{=} c\begin{pmatrix} a^2 \\ |b| \\ a \end{pmatrix} + \begin{pmatrix} a'^2 \\ |b'| \\ a' \end{pmatrix}$

$(ca+a')^2 \stackrel{?}{=} ca^2 + a'^2$ X

3. $D: P_n \to P_{n-1}$ $\quad \hat{x}: P_n \to P_{n+1}$

3(a)

$DF := \frac{dF}{dx}$

$D(x^3-7x) = 3x^2-7$

3(b)

$\hat{x}: P_n \to P_{n+1}$ (multiplication by x)

$\hat{x} \cdot F = x \cdot F$

$\hat{x}(7x) = 7x^2$

3(a) is linear $\quad D(f+g) \stackrel{?}{=} D(f) + D(g)$

$\Downarrow \qquad \Downarrow$

$(f+g)' = f' + g'$ ✓ (Prop 1 holds)

Proof 2

$(cF)' = c \cdot F'$ ✓.

hence differentiation is a linear trasformation.

(4) $A \in M_{m\times n}(F)$ $\quad A = (a_{ij}) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

Defined $L_A : F^n \longrightarrow F^m$

by $x = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots \; a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots \; a_{mn}x_n \end{pmatrix} = y.$

$y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$, $y_j = \sum_{i=1}^{n} a_{ji} x_i$ $\Leftarrow$ another way of writing y

L.H.S.

$$L(cx+z)_j = \sum_{i=1}^{n} a_{ji}(cx+z)_i$$
$$= \sum_{i=1}^{n} a_{ji}(cx_i + z_i)$$

RHS

$$[cL(x) + L(z)]_j = c\sum_{i=1}^{n} a_{ji}x_i + \sum_{i=1}^{n} a_{ji}z_i$$

LHS = RHS (Distributive law)