

Homework 6. TUT R 4-5PM.

Section 9.

$$3. \text{ a) } \begin{cases} G(x, y, u) = 0 \\ H(x, y, u) = 0 \end{cases} \Leftrightarrow \begin{cases} y = z, \\ G(x, y, u) = 0 \\ H(x, y, u) = 0 \end{cases} \Leftrightarrow (x, y, u) = (2, -1, 1)$$

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so consider $R\left(\begin{array}{c} y \\ u \end{array}\right) = \begin{pmatrix} G(x, y, u) \\ y \\ H(x, y, u) \end{pmatrix}$ $R: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \Leftrightarrow R\left(\begin{array}{c} y \\ u \end{array}\right) = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$ solvable using inverse functions

$$DR = \begin{pmatrix} \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial u} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial u} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial u} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & 2u \\ 0 & 1 & 0 \\ u & 9y^2 & x+3u^2 \end{pmatrix}$$

we want to define $x = g(y)$ & $u = h(y)$, so DR is invertible.

$$\Leftrightarrow \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & 2u \\ 0 & 1 & 0 \\ u & 9y^2 & x+3u^2 \end{pmatrix} \text{ is invertible} \Leftrightarrow \begin{pmatrix} \frac{\partial f}{\partial x} & 0 & 2u \\ 0 & 1 & 0 \\ u & 0 & x+3u^2 \end{pmatrix} = A \text{ is invertible.} \Leftrightarrow \det(A) \neq 0.$$

$$\det(A) = \frac{\partial f}{\partial x}(1(x+3u^2) + 2u(-u)) = \frac{\partial f}{\partial x}(x+3u^2) - 2u^2 = \frac{\partial f}{\partial x}(2+3(1)) - 2 \cdot 1^2 = 5 \frac{\partial f}{\partial x} - 2 \neq 0.$$

$\Rightarrow \boxed{\frac{\partial f}{\partial x} \neq \frac{2}{5}}$, then R is invertible near $(2, -1, 1)$ since $(x, y, u) = (2, -1, 1)$

then let $g(z) = \Pi_1(R^{-1}\left(\begin{array}{c} 0 \\ z \end{array}\right))$ $\Pi_1: \mathbb{R}^3 \rightarrow \mathbb{R}': \left(\begin{array}{c} y \\ u \end{array}\right) \mapsto x$ g, h are C' since R' is C'

$$h(z) = \Pi_3(R^{-1}\left(\begin{array}{c} 0 \\ z \end{array}\right)) \quad \Pi_3: \mathbb{R}^3 \rightarrow \mathbb{R}'': \left(\begin{array}{c} y \\ u \end{array}\right) \mapsto u \quad \& \Pi_1, \& \Pi_3 \text{ are } C'.$$

$$\text{so. } g(y) = \Pi_1(R'\left(\begin{array}{c} 0 \\ y \end{array}\right)) = \Pi_1\left(\begin{array}{c} y \\ u \end{array}\right) = x \Rightarrow g(-1) = 2$$

$$h(y) = \Pi_3(R'\left(\begin{array}{c} 0 \\ y \end{array}\right)) = \Pi_3\left(\begin{array}{c} y \\ u \end{array}\right) = u \Rightarrow h(-1) = 1$$

Therefore, when $\frac{\partial f}{\partial x} = D.f \neq \frac{2}{5}$ ensure that there are functions $x = g(y)$ & $u = h(y)$

such. $g(-1) = 2$ & $h(-1) = 1$.

b) from part a) $\begin{cases} G(g(y), y, h(y)) = 0 \\ H(g(y), y, h(y)) = 0 \end{cases}$:

at $y = -1$.

$$\Rightarrow \begin{cases} \left(\frac{\partial G}{\partial x} \frac{\partial G}{\partial y} \frac{\partial G}{\partial u}\right)\left(\begin{array}{c} g(y) \\ y \\ h(y) \end{array}\right) = 0 \\ \left(\frac{\partial H}{\partial x} \frac{\partial H}{\partial y} \frac{\partial H}{\partial u}\right)\left(\begin{array}{c} g(y) \\ y \\ h(y) \end{array}\right) = 0 \end{cases}$$

$$R \xrightarrow{y \mapsto \begin{pmatrix} g(y) \\ y \\ h(y) \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} y \\ u \end{pmatrix} \mapsto G\left(\begin{array}{c} y \\ u \end{array}\right)} R$$

$$\Rightarrow \begin{cases} \left(\frac{\partial f}{\partial x} \frac{\partial f}{\partial y} 2u\right)\left(\begin{array}{c} g'(y) \\ 1 \\ h'(y) \end{array}\right) = 0 \\ (u \ 9y^2 \ x+3u^2)\left(\begin{array}{c} g'(y) \\ 1 \\ h'(y) \end{array}\right) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x}g' + \frac{\partial f}{\partial y} + 2uh' = 0 \end{cases}$$

$$\Rightarrow \begin{cases} ug' + 9y^2 + (x+3u^2)h' = 0. \end{cases}$$

$$\Rightarrow \begin{cases} 1 \cdot g' + 1 \cdot 3 + 2 \cdot 1 \cdot h' = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 1 \cdot g' + 9 \cdot (-1)^2 + (2+3 \cdot 1^2)h' = 0 \\ 3h' + 12 = 0 \end{cases} \Rightarrow \begin{cases} g' = 11 \\ h' = -4 \end{cases}$$

Since $(x, y, u) = (2, -1, 1)$

$$Df(2, -1) = [1, -3]$$

$$\text{so. } \frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = -3$$

$$\text{hence } g(-1) = 11$$

$$\text{h}'(-1) = -4$$

6. Let $g: \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ s.t. $g(x_1, x_2) = f(x)$. $x_1 \in \mathbb{R}^k, x_2 \in \mathbb{R}^n, x \in \mathbb{R}^{k+n}, (x_1, x_2) = (x)$

$$\text{(i.e.) } x = (\underbrace{x_{11}, x_{12}, \dots, x_{1k}}_{x_1}, \underbrace{x_{21}, x_{22}, \dots, x_{2n}}_{x_2})$$

Since $f(a) = 0$. So $g(a_1, a_2) = 0$ - g is also C' since f is C'.

now consider $g(z, y) = 0 \Leftrightarrow \begin{cases} z = x \\ g(x, y) = 0 \end{cases} \quad x, y \in \mathbb{R}^k, y \in \mathbb{R}^n$

so with $H(y) = (g(x, y))$ $H: \mathbb{R}^{k+n} \rightarrow \mathbb{R}^{k+n} \Leftrightarrow H(y) = \begin{pmatrix} z \\ 0 \end{pmatrix}$ solvable.

H is C' since x is C' & $g(x, y)$ is C'.

$$DH = \begin{pmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_1}{\partial y} \\ \frac{\partial H_2}{\partial x} & \frac{\partial H_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial 0}{\partial x} & \frac{\partial 0}{\partial y} \end{pmatrix} = \begin{pmatrix} I_{k \times k} & 0 \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

Now consider $DH(a_1, a_2)$.

since $Df(a)$ has rank n & $Df(a) = Dg(a_1, a_2)$

so $Dg(a_1, a_2)$ has rank n .

assume $a = (\underbrace{a_{11}, a_{12}, \dots, a_{1k}}_{a_1}, \underbrace{a_{21}, a_{22}, \dots, a_{2n}}_{a_2})$

$$\text{so } Dg(a_1, a_2) = \left(\frac{\partial g}{\partial a_1}, \frac{\partial g}{\partial a_2} \right) = \left(\begin{array}{cccc} \frac{\partial g_1}{\partial a_{11}} & \dots & \frac{\partial g_1}{\partial a_{1k}} & \frac{\partial g_1}{\partial a_{21}} & \dots & \frac{\partial g_1}{\partial a_{2n}} \\ \vdots & & \vdots & & & \vdots \\ \frac{\partial g_n}{\partial a_{11}} & \dots & \frac{\partial g_n}{\partial a_{1k}} & \frac{\partial g_n}{\partial a_{21}} & \dots & \frac{\partial g_n}{\partial a_{2n}} \end{array} \right) \Big|_n$$

Since $Dg(a_1, a_2) \in M_{n \times (k+n)}$

& has rank n , then it has

a right inverse: an $(k+n) \times n$ matrix B s.t. $Dg(a_1, a_2) \cdot B = I_n \Rightarrow \left(\frac{\partial g}{\partial a_1}, \frac{\partial g}{\partial a_2} \right) \cdot B = I_n$.

so we can use elementary operations on $Dg(a_1, a_2)$ to make $\frac{\partial g}{\partial a_2} \in M_{n \times n}$ has rank n
then $\frac{\partial g}{\partial a_2}$ is invertible.

so $\begin{pmatrix} I_k & 0 \\ 0 & \frac{\partial g}{\partial a_2} \end{pmatrix}$ is invertible $\Leftrightarrow \begin{pmatrix} I_k & 0 \\ \frac{\partial g}{\partial a_1} & \frac{\partial g}{\partial a_2} \end{pmatrix}$ is invertible $\Leftrightarrow DH(a_1, a_2)$ is invertible.

so $\exists H^{-1}$ in some nbd of (a_1, a_2) . (i.e.) H is invertible near (a_1, a_2)

Since $H\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} g(a_1, a_2) \end{pmatrix}$ so both functions are invertible near (a_1, a_2)

$\Rightarrow g$ is invertible near (a_1, a_2) .

Since $g(x_1, x_2) = f(x)$, so f is invertible near a . & $f(a) = 0$.

By the property of invertible function. (let U be a nbd of a . & V be a nbd of $f(a) = 0$)

$\forall c \in V, \exists x \in U$ s.t. $f^{-1}(c) = x$. (i.e.) $f(x) = c$.

Therefore. if c is a point of \mathbb{R}^n sufficiently close to 0. (i.e. $c \in V$)

then the equation $f(x) = c$ has a solution.

Section 10.

1. Assume $P = \{P_1, \dots, P_n\}$ be the partition of Q . P_j is a partition of $[c_j, b_j]$

$$R \in P^V \Leftrightarrow R = \bigcap_{j=1}^n [c_j, d_j] \text{ s.t. } \forall j. [c_j, d_j] \in P_j$$

$$m_R(f) = \inf \{f(x) : x \in R\} \quad m_R(g) = \inf \{g(x) : x \in R\}$$

$$M_R(f) = \sup \{f(x) : x \in R\} \quad M_R(g) = \sup \{g(x) : x \in R\}$$

$$\exists x_0 \in R. \text{ s.t. } f(x_0) \leq f(x) \text{ for all } x \in R. \quad f(x_0) = m_R(f)$$

$$\text{Since } f(x) \leq g(x) \text{ for } x \in Q, \text{ so. } f(x_0) \leq f(x) \leq g(x) \text{ for all } x \in R.$$

$$\exists x'_0 \in R. \text{ s.t. } g(x'_0) \leq g(x) \text{ for all } x \in R. \quad g(x'_0) = M_R(g).$$

$$\text{Since } f(x_0) \leq g(x) \text{ for all } x \in R \text{ & } x'_0 \in R, \text{ so. } f(x_0) \leq g(x'_0)$$

$$\Rightarrow [m_R(f) \leq M_R(g)]$$

$$\exists x_1 \in R. \text{ s.t. } f(x_1) \geq f(x) \text{ for all } x \in R. \quad f(x_1) = M_R(f)$$

$$\text{since } x_1 \text{ s.t. } f(x_1) \leq g(x) \text{ for all } x \in Q, \text{ so. } f(x_1) \leq g(x_1)$$

$$\exists x'_1 \in R. \text{ s.t. } g(x'_1) \geq g(x) \text{ for all } x \in R. \quad g(x'_1) = M_R(g)$$

$$\text{since } x_1 \in R \text{ & } g(x'_1) \geq g(x_1), \text{ so. } g(x'_1) \geq g(x_1) \geq f(x_1)$$

$$\Rightarrow [M_R(f) \leq M_R(g)]$$

$$\text{Volume of } R: V(R) = V(\bigcap_{i=1}^n [c_i, d_i]) = \sum_{i=1}^n (d_i - c_i).$$

$$L(f, P) = \sum_{R \in P} V(R) m_R(f) \leq \sum_{R \in P} V(R) m_R(g) = L(g, P)$$

$$U(f, P) = \sum_{R \in P} V(R) M_R(f) \leq \sum_{R \in P} V(R) M_R(g) = U(g, P)$$

$$\int_Q f = \sup_{P \in \mathcal{P}} L(f, P) \leq \sup_{P \in \mathcal{P}} L(g, P) = \int_Q g. \quad \left\{ \begin{array}{l} f \leq g \\ \int_Q f \leq \int_Q g \end{array} \right.$$

$$\int_Q f = \inf_{P \in \mathcal{P}} U(f, P) \leq \inf_{P \in \mathcal{P}} U(g, P) = \int_Q g. \quad \left\{ \begin{array}{l} f \leq g \\ \int_Q f \leq \int_Q g \end{array} \right.$$

Problem A. $f(x) = \sin(\frac{1}{x})$ $x \in (0, 1]$ is bounded cont. but not uniformly cont.

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Since $\sin(x) \in [-1, 1]$ for all $x \in \mathbb{R}$. So $f(x)$ is bounded. $f(x) \in [-1, 1]$.

Since \sin is cont. on \mathbb{R} & $\frac{1}{x}$ is cont. on $(0, 1]$. So $f(x)$ is cont. on $(0, 1]$.

Want to show $f(x)$ is not uniformly cont.

Let $\epsilon = \frac{1}{2}$, & assume f is uniformly cont.

st. $\forall x, y \in \mathbb{R}$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

Consider $\begin{cases} \frac{1}{x} = 2k\pi + \frac{\pi}{2} \\ \frac{1}{y} = 2k\pi + \frac{3\pi}{2} \end{cases}$ where $k \in \mathbb{Z}^+$ $x = \frac{1}{2k\pi + \frac{\pi}{2}}$ $y = \frac{1}{2k\pi + \frac{3\pi}{2}}$ $x, y \in (0, 1]$ Since both $\frac{1}{x}$ & $\frac{1}{y} > 1$.

whatever how small δ is, we can choose a k larger enough to make $|x-y| < \delta$.

but $|f(x) - f(y)| = |\sin(\frac{1}{x}) - \sin(\frac{1}{y})| = |\sin(2k\pi + \frac{\pi}{2}) - \sin(2k\pi + \frac{3\pi}{2})| = ||1 - (-1)|| = 2 > \frac{1}{2} = \epsilon$

$\Rightarrow |f(x) - f(y)| > \epsilon$. Contradiction

Therefore, $f(x) = \sin(\frac{1}{x})$ is not uniformly cont.