Fri. 3rd Mar. Hour 058 (Read along: 29-31

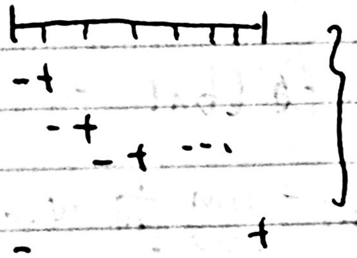
Today: $\int_M dw = \int_{\partial M} w$ $\quad d: \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ by

$$dw(\xi_1, \cdots, \xi_{k+1}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{k+1}} w(\partial P(\varepsilon \xi_1, \cdots, \varepsilon \xi_{k+1}))$$

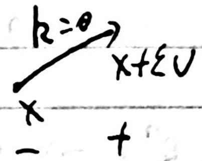
where $P(\xi_1, \cdots, \xi_{k+1})$  $\xi_i = (x, v_i)$

$$\Omega^0(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3)$$

$$\left\{\begin{matrix} a_1(x)\, dx_1 \\ a_2(x)\, dx_2 \\ a_3(x)\, dx_3 \end{matrix}\right\} \to \left(\begin{matrix} b_1(x)\, dx_2 \wedge dx_3 \\ b_2(x) dx_3 \wedge dx_1 \\ b_3(x)\, dx_1 \wedge dx_3 \end{matrix}\right) \quad (c(x)\, dx_1 \wedge dx_2 \wedge dx_3)$$



$\to$ on functions.



$$(dF)(\xi) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F(x+\varepsilon v) - F(x))$$
$$= D_\xi F = DF_x \cdot v$$

Aside: $x_j : \mathbb{R}^n \to \mathbb{R}$, "the coordinate functions on $\mathbb{R}^n$"

$$(dx_j)(x, e_i) = \frac{\partial x_j}{\partial x_i} = \delta_{ij} = \phi_j(x, e_i) \Rightarrow dx_j = \phi_j$$

Claim: If $F \in C^1(\mathbb{R}^n)$.

$$dF = \sum_i \frac{\partial F}{\partial x_i} \phi_i = \sum_i \frac{\partial F}{\partial x_i} dx_i$$

On $\mathbb{R}^1$. $dF = \frac{df}{dx} \cdot dx$

Pf: evaluate both sides on (x, e_j).

$$LHS = dF(x, e_j) = D_{e_j} F(x) = \frac{\partial F}{\partial x_j}$$

$$RHS = \sum_i \frac{\partial F}{\partial x_i}(x) (dx_i)(e_j) = \sum_i \frac{\partial F}{\partial x_i} \delta_{ij}$$

$$\Omega^0 \rightarrow \Omega^1 \rightarrow \Omega^2 \rightarrow \Omega^3$$

$$\text{functions} \xrightarrow[\nabla \cdot F]{\text{grad}(F)} \text{v.f.} \xrightarrow[\nabla \times]{\text{curl (next class)}} \text{v.f.} \xrightarrow[\nabla \cdot B]{\text{div}(B) \text{ (next class)}} F$$

$$F \rightarrow \frac{\partial F}{\partial x_1} dx_1 + \frac{\partial F}{\partial x_2} dx_2 + \frac{\partial F}{\partial x_3} dx_3 \iff \begin{Bmatrix} \frac{\partial F}{\partial x_1} \\ \frac{\partial F}{\partial x_2} \\ \frac{\partial F}{\partial x_3} \end{Bmatrix}$$

Tmr: $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = B$, $\sum \frac{\partial b_i}{\partial x_i}$

Thm: $\exists !$ linear operator $d: \Omega^k(\mathbb{R}^n) \rightarrow \Omega^{k+1}(\mathbb{R}^n)$ Such that

1. if $F \in \Omega^0(\mathbb{R}^n)$, then $dF = \sum_i \frac{\partial F}{\partial x_i} dx_i$
2. $\begin{pmatrix} w \in \Omega^k \\ \eta \in \Omega^l \end{pmatrix}$ $d(w \wedge \eta) = (dw) \wedge \eta + (-1)^k w \wedge (d\eta)$
3. $d^2 = 0$. $d \cdot d = 0$ If $w \in \Omega^k(\mathbb{R}^n)$, then $d(dw) = 0$ in $\Omega^{k+2}(\mathbb{R}^n)$

$$dw = d\Big(\sum_{I \in \binom{n}{k}} a_I(x)\, dx_I\Big) \qquad (dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k})$$

$$= \sum_I d(a_I(x) dx_I) = \sum_I (da_I) \wedge (dx_I) \pm \sum a_I \wedge d(dx_I)$$

↳ plus, since a is a function

$$= \sum_I \Big(\sum_i \frac{\partial a_I}{\partial x_i}\, dx_i\Big) \wedge dx_I + 0 \quad \big(d(dx_I) = d(\overset{dx_{i_1}}{\cancel{d}} \wedge \cdots \wedge dx_{i_k}) = 0\big)$$

$$= \sum_I \frac{\partial a_I}{\partial x_i}\,(dx_i \wedge dx_I)$$

$\Rightarrow$ We have proven ~~on~~ uniqueness in thm!

Ex:

$$d(b_1\, dx_2 \wedge dx_3 + b_2 \cdots)$$

$$= (db_1) \wedge dx_2 \wedge dx_3 + \cdots$$

$$= \Big(\frac{\partial b_1}{\partial x_1} dx_1 + \frac{\partial b_2}{\partial x_2} dx_2 + \frac{\partial b_3}{\partial x_3} dx_3\Big)\, dx_2 \wedge dx_3 + \cdots$$

$$=0 \qquad =0$$

$$= \frac{\partial b_1}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial b_2}{\partial x_2} dx_2 \wedge dx_3 \wedge dx_1 + \cdots$$

$$= \Big(\frac{\partial b_1}{\partial x_1} dx_1 + \frac{\partial b_2}{\partial x_2} dx_2 + \frac{\partial b_3}{\partial x_3} dx_3\Big)\, dx_{123} \rightsquigarrow dx_I.$$