MAT257 Midterm I Review

LINEAR ALGEBRA & MATRICES

Inner products: Given x, y vectors in vector space V, the inner product $\langle x, y \rangle \in \mathbb{R}$ such that; for $x, y, z \in V$, $c \in \mathbb{R}$:

① $\langle x, y \rangle = \langle y, x \rangle$ ② $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

③ $\langle cx, y \rangle = c\langle x, y \rangle = \langle x, cy \rangle$ ④ $\langle x, x \rangle > 0$ if $x \neq 0$.

Ex: $\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$ (sometimes called dot product).

DOT PRODUCT

If V is an inner product space, the length/norm of a vector of V:

$$\|x\| = \langle x, x \rangle^{1/2}.$$ EUCLIDEAN NORM

Norm properties: ① $\|x\| > 0$ if $x \neq 0$ ② $\|cx\| = |c| \cdot \|x\|$

③ $\|x+y\| \leq \|x\| + \|y\|$

③' $\|x-y\| \geq \|x\| - \|y\|$

There is another useful norm defined as follows:

$$|x| = \max\{|x_1|, \dots, |x_n|\}$$ SUP NORM

Sup norm and euclidean norm satisfy the following inequality:

$$|x| \leq \|x\| \leq \sqrt{n}\,|x|$$ NORM RELATION

Cauchy-Schwarz Inequality: $\langle u, v \rangle \leq \|u\| \cdot \|v\|$

Matrices: $A \in M_{n \times m}(\mathbb{R}) \Rightarrow$ matrix of reals with n rows, m columns.

If $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$:

① $A + B = (a_{ij} + b_{ij})$

② $KA = (ka_{ij})$

③ $C = A \cdot B \Rightarrow c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$

④ $|A| = \max(a_{ij})$, $i = 1, \dots, n$ and $j = 1, \dots, m$.

MATRIX SUP NORM

Also: ① $A \cdot (B \cdot C) = (A \cdot B) \cdot C$

② $A \cdot (B + C) = A \cdot B + A \cdot C$

③ $(A+B) \cdot C = A \cdot C + B \cdot C$

④ $(cA) \cdot B = c(A \cdot B) = A \cdot (cB)$

⑤ $I_n \cdot A = A$, $A \cdot I_m = A$

IDENTITY MATRIX

THM 1.3 If $A_{n\times m}$ and $B_{m\times p}$, then $|A\cdot B| \le m|A|\cdot|B|$. MATRIX SUP INEQUALITY

Determinants: If $A_{n\times n}$; $B_{n\times n}$: ① $\det(A\cdot B) = \det(A)\cdot\det(B)$.

② $\det(A^T) = \det(A)$ ③ $\det A = b(-1)^{i+j}\det A_{ij}$

all entries in row/col except b go to 0.

THM 2.13 Let $A_{n\times n} = [a_1 \cdots a_n]$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, s.t. $Ax = c$,

then $(\det A)\cdot x_i = \det[a_1 \cdots a_{i-1}\ c\ a_{i+1} \cdots a_n]$. CRAMER'S RULE.

THM 2.14 Let $A_{n\times n}$ matrix of rank n; $B = A^{-1}$, then:

$$b_{ij} = \frac{(-1)^{i+j}\det A_{ji}}{\det A}.$$

COFACTOR EXPANSION

THM 2.15 Let $A_{n\times n}$ and i fixed: $\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ik}\cdot\det A_{ik}$.