Section 9.

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3. $f: \mathbb{R}^2 \to \mathbb{R}$ of class C^1, $f(2,-1) = -1$. $G(x,y,u) = f(x,y) + u^2$, $H(x,y,u) = ux + 3y^3 + u^3$

$G(x,y,u) = 0$ & $H(x,y,u) = 0$ $\Rightarrow$ $(x,y,u) = (2,-1,1)$.

a) Condition of Df ensure $x = g(y)$ & $u = h(y)$ exist. of C^1, on open set in $\mathbb{R}$, satisfy $g(-1)=2$ $h(-1)=1$.

Sol^n: Define $F\begin{pmatrix} x \\ y \\ u \end{pmatrix} = \begin{pmatrix} G(x,y,u) \\ y \\ H(x,y,u) \end{pmatrix}$

then $DF = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & 2u \\ 0 & 1 & 0 \\ u & 9y^2 & x+3u^2 \end{pmatrix}$ $\det(DF)(2,-1) = (x+3u^2)\frac{\partial f}{\partial x} - 2u^2 = 5\frac{\partial f}{\partial x}(2,-1) - 2 \neq 0$.

need $\frac{\partial f}{\partial x}(2,-1) \neq \frac{2}{5}$

then $F(2,-1,1) = (0,-1,0)$ and applying the inverse F.T. to F we can conclude:

$\exists$ open set $U \times V$ of $\mathbb{R}^3$ about $(2,-1,1)$ where U open in $\mathbb{R}^2$ and V open in $\mathbb{R}$ satisfy:

i) F maps $U \times V$ in a 1-1 fashion onto an open set W in $\mathbb{R}^3$ containing $(0,-1,0)$

ii) $F^{-1}: W \to U \times V$ is of class C^1, then we have $(x,y,u) = F^{-1}(G(x,y,u), y, H(x,y,u))$

Because both F & F^{-1} perserves the 1st & 3rd coordinates $\Rightarrow$ then can write $F^{-1}(z_1, y, z_2) = (g_0(x,y,u), y, h_0(x,y,u))$

For a small enough nbd of -1 in $\mathbb{R}$, say B, we have:

$F^{-1}(0,y,0) = (g_0(0,y,0), y, h_0(0,y,0))$

$(0,y,0) = F(g_0(0,y,0), y, h_0(0,y,0)) = \left(G\begin{pmatrix} g_0(0,y,0) \\ y \\ h_0(0,y,0) \end{pmatrix}, y, H\begin{pmatrix} g_0(0,y,0) \\ y \\ h_0(0,y,0) \end{pmatrix}\right)$

$\Rightarrow G(g_0, y, h_0) = 0$ and $H(g_0, y, h_0) = 0$ at $(0,y,0)$.

Set $g(y) = g_0(0,y,0)$ and $h(y) = h_0(0,y,0)$ for $y \in B$.

then g, h satisfies $G(g(y), y, h(y)) = 0$ and $H(g(y), y, h(y)) = 0$.

$\Rightarrow (g(y), y, h(y)) = (2,-1,1)$ i.e. $g(-1) = 2$ and $h(-1) = 1$ $\square$.

b). Under condition of a). and $Df(2,-1) = (1 \quad -3)$. Find $g'(-1)$ and $h'(-1)$.

sol^n: $0 = G(g(y), y, h(y))$ $0 = H(g(y), y, h(y))$ take D on both sides then:

$$\begin{cases} 0 = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & 2u \end{pmatrix}\begin{pmatrix} g'(y) \\ 1 \\ h'(y) \end{pmatrix} = g'(y)\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + 2uh'(y) \\ 0 = \begin{pmatrix} u & 9y^2 & 3u^2+x \end{pmatrix}\begin{pmatrix} g'(y) \\ 1 \\ h'(y) \end{pmatrix} = ug'(y) + 9y^2 + (3u^2+x)h'(y) \end{cases}$$

holds at $(x,y,u) = (2,-1,1)$ $\Rightarrow$ $\begin{cases} g'(-1) - 3 + 2h'(-1) = 0 \\ g'(-1) + 9 + 5h'(-1) = 0 \end{cases}$ $\Rightarrow$ $\begin{cases} g'(-1) = 11 \\ h'(-1) = -4 \end{cases}$ $\square$

$\frac{\partial f}{\partial x}(2,-1) = 1$ $\frac{\partial f}{\partial y}(2,-1) = -3$

6. $f: \mathbb{R}^{k+n} \to \mathbb{R}^n$ of class C'; $f(a)=0$ and $Df(a)$ has rank n.

Thm: if c is a pt of $\mathbb{R}^n$ sufficiently close to 0, then $f(x)=c$ has a solution.

proof: Suppose that $Df(a) = \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_{n+k} \\ \vdots & & \vdots \\ \partial f_n/\partial x_1 & \cdots & \partial f_n/\partial x_{n+k} \end{pmatrix}$ $\text{rank}(Df(a)) = n$

$\Rightarrow$ we can find $\{y_1, \cdots, y_n\} \subseteq \{x_1, \ldots, x_{n+k}\}$ s.t. $\{\partial f/\partial y_1, \cdots \partial f/\partial y_n\}$ Linearly independent.

Let $\{z_1, \cdots, z_k\} = (\{x_1, \ldots, x_{n+k}\} \setminus \{y_1, \cdots, y_n\})$ then $\partial f/\partial z_i = \sum_{j=1}^{n} b_{ij} (\partial f/\partial y_j)$ for every $i: 1 \le i \le k$

We can write $f(x) = f(z, y)$ where $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^k$

We have $\det\left(\frac{\partial f}{\partial y}\right) \neq 0$ as $\{\partial f/\partial y_1, \ldots \partial f/\partial y_n\}$ are L.I. Also have $f(e,d) = 0$ for $\binom{e}{d} = a$, $e \in \mathbb{R}^k$, $d \in \mathbb{R}^n$

Define $F(z,y) = F(z, f(z,y))$, then F maps an open set A of $\mathbb{R}^{k+n}$ into $\mathbb{R}^k \times \mathbb{R}^n = \mathbb{R}^{k+n}$

$DF = \begin{pmatrix} I_k & 0 \\ \partial f/\partial z & \partial f/\partial y \end{pmatrix}$ $\det(DF) = \det(\partial f/\partial y) \neq 0$ at $a = \binom{e}{d}$

Now $F(e,d) = (e,0)$ we can apply inverse function theorem on the map F.

Conclude that $\exists$ an open set $U \times V$ of $\mathbb{R}^{k+n}$ about (e,d) where $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^n$ so that

i) F maps $U \times V$ in a 1-1 fashion onto an open set W in $\mathbb{R}^{k+n}$ containing $(e,0)$

ii) its inverse $G: W \to U \times V$ is of class C'

Because $F(z,y) = (z, f(z,y))$ we have $(z,y) = F^{-1}(z, f(z,y)) = G(z, f(z,y))$

~~i.e. if c is a pt of $\mathbb{R}^n$ sufficient close to 0 that we have $(z,c) \in W$~~, ~~Calculate $G(z,$~~

Thus G perserves the first k coordinates, as F does, then can write G in the form

$G(z,p) = (z, h(z,p))$ for $z \in \mathbb{R}^k$ and $p \in \mathbb{R}^n$; here h is a function mapping $W \to \mathbb{R}^n$ of class C'

i.e. for every c of $\mathbb{R}^n$ sufficiently close to 0 so that $(z,c) \in W$,

calculate $G(z,c) = (z, h(z,c))$, then $F(G(z,c)) = F(z, h(z,c))$

i.e. take $y = h(z,c)$ so $F(z,y) = F(G(z,c)) = (z,c)$, here $c = f(z,y) = f(x)$.

~~Section~~. Addition Ques.

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A. Example of a bounded. cont. function on $(0,1]$ which is not uniformly cont.

Sol^n: Take $f(x) = \sin(\frac{1}{x})$. it is clearly bdd on $(0,1]$ by -1 and 1.

- $f(x)$ is continuous on $(0,1]$ because $\frac{1}{x}$ cont. on $(0,1]$ and $\sin y$ cont. on $(0,1]$.
- $f(x)$ is not unif. cont. on $(0,1]$.

Given $\varepsilon = \frac{1}{4}$, For every $\delta > 0$, can find positive $n > \frac{\sqrt{1+8\delta\pi}-1}{4}$ so that $n(2n+1) > \delta\pi$.

that is, $\left|\frac{2}{(2n+1)\pi} - \frac{1}{n\pi}\right| < \delta$. where $f(\frac{2}{(2n+1)\pi}) = \sin(\frac{(2n+1)\pi}{2}) = \pm 1$, $f(\frac{1}{n\pi}) = \sin(n\pi) = 0$

Letting $x = \frac{2}{(2n+1)\pi}$ and $y = \frac{1}{n\pi}$. we have $|x-y| < \delta$ but $|f(x) - f(y)| = |\pm 1 - 0| = 1 > \varepsilon$ $\square$

Section 10.

1. Let $f, g: Q \to \mathbb{R}$ bdd, $f(x) \le g(x)$ for $x \in Q$. Show that $\underline{\int}_Q f \le \underline{\int}_Q g$ and $\overline{\int}_Q f \le \overline{\int}_Q g$.

proof: $\underline{\int}_Q f = \sup_P L(f,P) = \sup_P \sum_{R \in P} V(R) \cdot m_R(f)$ where $P = (P_1, \dots, P_n)$ $m_R(f) = \inf\{f(x) : x \in R\}$.

$\overline{\int}_Q f = \inf_P U(f,P) = \inf_P \sum_{R \in P} V(R) \cdot M_R(f)$ $V(R) = \prod_{j=1}^{n} (d_j - c_j)$ $M_R(f) = \sup\{f(x) : x \in R\}$.

- For given partition P_0, we have $L(f, P_0) = \sum_{R \in P_0} V(R) \cdot m_R(f) \le \sum_{R \in P_0} V(R) \cdot m_R(g) = L(g, P_0)$ because partition for f & g same there, then for same $R \in P$, always have $\inf_{x \in R} f(x) \le \inf_{x \in R} g(x)$. And we have $L(g, P_0) \le \sup_P L(g, P_0)$, where the right hand side is independent of P chosen. from above $L(f,P) \le L(g,P) \le \sup_P L(g,P)$ so $L(f,P) \le \sup_P L(g,P)$ for any partition P. take sup over P both sides then $\sup_P L(f,P) \le \sup_P L(g,P)$ as RHS independent of P. i.e. $\underline{\int}_Q f \le \underline{\int}_Q g$.
- Similarly we have $U(f,P) = \sum_{R \in P} V(R) \cdot M_R(f) \le \sum_{R \in P} V(R) \cdot M_R(g) = U(g,P)$ when P same for f & g ~~then~~ we have $U(f,P) \le U(g,P)$ and $\inf_P U(f,P) \le U(f,P)$ where LHS is indep. of P therefore $\inf_P U(f,P) \le U(g,P)$ for any partition, so $\inf_P U(f,P) \le \inf_P U(g,P)$. i.e. $\overline{\int}_Q f \le \overline{\int}_Q g$.