Problem Set 17 — MAT257

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Disclaimer—This page has been typeset by a student as a *convenient* consolidation of the homework problems. There inevitably will be mistakes; always defer to the official handout!

Problems marked with $*$ are to be submitted for credit.

1 Munkres §32 (pp.273–274)

1. Prove Theorem 32.1 when ω and η have order zero and when θ has order zero.

2. Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^6$ be a $\mathcal{C}^\infty$ map. Show directly that

$$d\alpha_1 \wedge d\alpha_3 \wedge d\alpha_5 = \big(\det D\alpha(1,3,5)\big)\, dx_1 \wedge dx_2 \wedge dx_3.$$

$*$ 3. In $\mathbb{R}^3$, let

$$\omega = xy\, dx + 2z\, dy - y\, dz.$$

Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the equation

$$\alpha(u,v) = (uv, u^2, 3u+v).$$

Calculate $d\omega$ and $\alpha^*\omega$ and $\alpha^*(d\omega)$ and $d(\alpha^*\omega)$ directly.

4. Show that (a) of Theorem 32.2 is equivalent to the formula $\alpha^*(dy_i) = d(\alpha^* y_i)$, where $y_i : \mathbb{R}^n \to \mathbb{R}$ is the i^{th} projection function in $\mathbb{R}^n$.

$*$ 5. Prove the following formula for computing $\alpha^*\omega$ in general:

Theorem. Let A be open in $\mathbb{R}^k$; let $\alpha : A \to \mathbb{R}^n$ be of class $\mathcal{C}^\infty$. Let $\mathbf{x}$ denote the general point of $\mathbb{R}^k$; let $\mathbf{y}$ denote the general point of $\mathbb{R}^n$. If $I = (i_1, \ldots, i_l)$ is an ascending l-tuple from the set $\{1, \ldots, n\}$, then

$$\alpha^*(dy_I) = \sum_{[J]} (\det \frac{\partial \alpha_I}{\partial x_J})\, dx_J.$$

Here $J = (j_1, \ldots, j_l)$ is an ascending l-tuple from the set $\{1, \ldots, k\}$ and

$$\frac{\partial \alpha_I}{\partial x_J} = \frac{\partial(\alpha_{i_1}, \ldots, \alpha_{i_l})}{\partial(x_{j_1}, \ldots, x_{j_l})}.$$

6. This exercise shows that the transformations α_i and β_j of §31 do not in general behave well with respect to the maps induced by a diffeomorphism α.

   Let $\alpha : A \to B$ be a diffeomorphism of open sets in $\mathbb{R}^n$. Let $\mathbf{x}$ denote the general point of A and let $\mathbf{y}$ denote the general point of B. If $F(\mathbf{x}) = \big(\mathbf{x}; f(\mathbf{x})\big)$ is a vector field in A, let $G(\mathbf{y}) = \alpha_*\big(F(\alpha^{-1}(\mathbf{y}))\big)$ be the corresponding vector field in B.

   (a) Show that the 1-forms $\alpha_1 G$ and $\alpha_1 F$ do not in general correspond under the map α^*. Specifically, show that $\alpha^*(\alpha_1 G) = \alpha_1 F$ for all F if and only if $D\alpha(\mathbf{x})$ is an orthogonal matrix for each $\mathbf{x}$.
   [*Hint:* Show that the equation $\alpha^*(\alpha_1 G) = \alpha_1 F$ is equivalent to the equation

   $$D\alpha(\mathbf{x})^T \cdot D\alpha(\mathbf{x}) \cdot f(\mathbf{x}) = f(\mathbf{x}).\ ]$$

   (b) Show that $\alpha^*(\beta_{n-1} G) = \beta_{n-1} F$ for all F if and only if $\det D\alpha = +1$.
   [*Hint:* Show that the equation $\alpha^*(\beta_{n-1} G) = \beta_{n-1} F$ is equivalent to the equation

   $$f(\mathbf{x}) = \big(\det D\alpha(\mathbf{x})\big) \cdot f(\mathbf{x}).\ ]$$

   (c) If h is a scalar field in A, let $k = h \circ \alpha^{-1}$ be the corresponding scalar field in B. Show that $\alpha^*(\beta_n k) = \beta_n h$ for all h if and only if $\det D\alpha = +1$.

7. Use Exercise 6 to show that if α is an orientation-preserving isometry of $\mathbb{R}^n$, then the operator $\widetilde{\alpha}_*$ on vector fields and scalar fields commutes with the operators grad and div, and with curl if $n = 3$.

2 Munkres §33 (p.280)

1. Let $A = (0,1)^2$. Let $\alpha : A \to \mathbb{R}^3$ be given by the equation

   $$\alpha(u, v) = (u, v, u^2 + v^2 + 1).$$

   Let Y be the image set of α. Evaluate the integral over Y_α of the 2-form

   $$x_2\, dx_2 \wedge dx_3 + x_1 x_3\, dx_1 \wedge dx_3.$$

* 2. Let $A = (0,1)^3$. Let $\alpha : A \to \mathbb{R}^4$ be given by the equation

   $$\alpha(s, t, u) = \big(s, u, t, (2u - t)^2\big).$$

   Let Y be the image set of α. Evaluate the integral over Y_α of the 3-form

   $$x_1\, dx_1 \wedge dx_4 \wedge dx_3 + 2x_2 x_3\, dx_1 \wedge dx_2 \wedge dx_3.$$

* 3. Let A be the open unit ball in $\mathbb{R}^2$. Let ω be the 2-form

   $$(1/\|\mathbf{x}\|^m)(x_1\, dx_2 \wedge dx_3 - x_2\, dx_1 \wedge dx_3 + x_3\, dx_1 \wedge dx_2).$$

   Let $\alpha_1, \alpha_2 : A \to \mathbb{R}^3$ be given by the equations

   $$\alpha_1(u, v) = (u, v, \sqrt{1 - u^2 - v^2}),$$
   $$\alpha_2(u, v) = (u, v, -\sqrt{1 - u^2 - v^2}).$$

   (a) Let Y be the image set of α_1. Evaluate the integral over Y_{α_1} of ω.

   (b) Let Y be the image set of α_2. Evaluate the integral over Y_{α_2} of ω.

4. If η is a k-form in $\mathbb{R}^k$, and if $\mathbf{a}_1, \ldots, \mathbf{a}_k$ is a basis for $\mathbb{R}^k$, what is the relation between the integrals

   $$\int_A \eta \qquad \text{and} \qquad \int_{\mathbf{x} \in A} \eta(\mathbf{x})\big((\mathbf{x}; \mathbf{a}_1), \ldots, (\mathbf{x}; \mathbf{a}_k)\big) \qquad ?$$

   Show that if the frame $(\mathbf{a}_1, \ldots, \mathbf{a}_k)$ is orthonormal and right-handed, then they are equal.

3 "Ponder..."

Challenge! What was it that you computed, in problem 3 of §33? Could you have done it without any actual computation?