19th Fri. Mar. hour 061

Read along: 30–32. Office hours: Drer Mon 5–8

Jeff Tue 11–2

$\phi := \mathbb{R}^n \to \mathbb{R}^m \Rightarrow \phi^* := \Omega^k(\mathbb{R}^m) \to \Omega^k(\mathbb{R}^n)$

1. ϕ^* linear
2. $(\phi \circ \psi)^* = \psi^* \circ \phi^*$
3. $\phi^*(w \wedge \eta) = \phi^*(w) \wedge \phi^*(\eta)$
4. $\phi^*(dw) = d\phi^*(w)$ → owned.

$\left.\begin{matrix}1.\\2.\\3.\end{matrix}\right\}$ done

$$\int_M dw = \int_{\partial M} w$$

Ex: $\mathbb{R}^2_{r\theta} \xrightarrow{\phi} \mathbb{R}^2_{xy}$, $\begin{pmatrix} r \\ \theta \end{pmatrix} \to \begin{pmatrix} r\sin\theta \\ r\cos\theta \end{pmatrix}$ $\quad W = \dfrac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2_{xy})$

$\phi^*(x) = r\cos\theta,\ \phi^*(y) = r\sin\theta)$

Pull back every parts:

$\phi^* w: \dfrac{r\cos\theta\, d(r\sin\theta) - r\sin\theta\, d(r\cos\theta)}{(r\cos\theta)^2 + (r\sin\theta)^2} = \#$

$d(r\cos\theta) = \dfrac{\partial(r\cos\theta)}{\partial r} dr + \dfrac{\partial(r\cos\theta)}{\partial\theta} d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$

$d(r\sin\theta) = \sin\theta\, dr + r\cos\theta\, d\theta$

so $\# = \dfrac{r^2 d\theta}{r^2} = d\theta$. so ϕ^* is exact, so $d(\phi^* w) = 0$

$dw = d\left(\dfrac{x\,dy - y\,dx}{x^2+y^2}\right) = dx \wedge \left(\dfrac{y^2-x^2}{(x^2+y^2)^2} dy + \text{mess}\, dx\right.$

$\left. + dy \wedge (\text{mess}\, dy + \dfrac{y^2-x^2}{(x^2+y^2)^2} dx\right)$

$= \text{ugly}(dx \wedge dy) + \text{ugly}(dy \wedge dx) = 0$

$dw = \Sigma\, dx_j \wedge \dfrac{dw}{dx_j}$

$= dx \wedge \dfrac{\partial w}{\partial x} + dy \wedge \dfrac{\partial w}{\partial y}$

$\partial\left(\dfrac{x}{x^2+y^2}\right) dx = \dfrac{x^2+y^2 - x \cdot 2x}{(x^2+y^2)^2}$

$= \dfrac{y^2 - x^2}{(x^2+y^2)^2}$

$\partial\left(\dfrac{-y}{x^2+y^2}\right) dy$

$= \dfrac{-x^2-y^2}{(x^2+y^2)^2} = \text{ugly}$

Pf of 4: IF $w = F \in \Omega^0(\mathbb{R}^m)$.

$d(\phi^* F)(\xi) = D_\xi(\phi^* F) \overset{?}{=}$ $\xi \in T_x\mathbb{R}^n = (x, v)$, $x \in \mathbb{R}^n$

$\phi^*(dF)(\xi) = (dF)(\phi_* \xi) = (D_{\phi_* \xi})F$
$= D_{(\phi(x),\, D\phi_x v)} F$
$= DF_{\phi(x)} \cdot (D\phi_x \cdot v)$

$D(F \circ \phi)_x \cdot v$ $=$

Claim

~~Pf of 4~~

Both $\phi^*(dw)$ and $d(\phi^* w)$ are linear in w
so enough to consider $w = a \cdot dx_I = a \cdot dx_2 \wedge dx_5 \wedge dx_7$
↳ function

$d\phi^* w = d(\phi^*(a) \cdot \phi^*(dx_2) \wedge \phi^*(dx_5) \wedge \phi(dx_7))$
$= d(\phi^*(a) \cdot d(\phi^* x_2) \wedge d(\phi^* x_5) \wedge d(\phi^* x_7))$ 2 chainrule
$= (d(\phi^*(a)) \wedge d(\phi^* x_2) \wedge d(\phi^* x_5) \wedge d(\phi^* x_7) + 0$

$\Vert$

$\phi^*(dw) = \phi^*(da \wedge dx_2 \wedge dx_5 \wedge dx_7 + 0)$
$= d\phi^* a \wedge d\phi^* x_2 \wedge d\phi^* x_5 \wedge d\phi^* x_7)$

$\phi : \mathbb{R}^n_x \to \mathbb{R}^n_y$: $w \in \Omega^{top}(\mathbb{R}^n y) = F dy_{\underline{n}}$
$W = F \cdot (dy_1 \wedge \cdots \wedge dy_n) = F\, dy_{(1,2,3,\dots,n)} = F dy_{\underline{n}}$
Claim: $\phi^*(w) = \phi^*(F dy_{\underline{n}})$ → no abs value
$= \phi^* F \cdot J_\phi(x) \cdot dx_{\underline{n}}$
where $J_\phi(x) = \det\left(\frac{d\phi^i}{dx_j}\right)$
$= \det(D\phi_x)$

Ex: using claim for Ex above: $\phi^*(dx \wedge dy) = d(r\cos\theta) \wedge d(r\sin\theta)$
$= \cdots$

Claim: $r \cdot dr \wedge d\theta$