Problem 2. Let G be a group and let $Z(G)$ denote its center.

1. Show that if $G/Z(G)$ is cyclic then G is Abelian.

Let $xZ(G)$ be the generator for $Z(G)$. Since $Z(G) \triangleleft G$ (obvious) we know from understanding cosets that any $g \in G$ can be written as $x^m z$ for some $x^m \in G$ and $z \in Z(G)$.

Therefore for all $g_1 g_2 \in G$, $g_1 g_2 = (x^a z_1)(x^b z_2) = x^a z_1 x^b z_2 = x^{a+b} z_1 z_2$ (since $z_1 \in Z(G)$) $= x^b x^a z_2 z_1 = x^b z_2 x^a z_1 = g_2 g_1$ for all $g_1, g_2 \in G$ and therefore G is Abelian.$\square$

2. Prove that if the group $Aut(G)$ of automorphisms of G is cyclic, then G is Abelian.

Since $Inn(G) \prec Aut(G)$, $Inn(G)$ must also be cyclic. Now we will prove that $Inn(G) \approxeq G/Z(G)$. First let $\phi : G \to Inn(G)$ by $\phi_g =$ inner automorphism $G \to G$ by g. ϕ is clearly a homomorphism because for $a, b \in G$, $\phi_{ab} = abGb^{-1}a^{-1} = \phi_a(\phi_b)$. $Ker\phi$ is clearly equal to $Z(G)$. And by the first isomorphism theorem we know that $G/Ker\phi \approxeq im\phi \to G/Z(G) \approxeq Inn(G)$ and by the properties of isomorphisms we know that if $Inn(G)$ is cyclic then $G/Z(G)$ is also cyclic and in Part 1 we proved that if $G/Z(G)$ is cyclic then G is Abelian. Therefore G is Abelian.$\square$