Thm: A subset $\beta \subset V$ is a basis iff every $v \in V$ can be expressed as a lin. comb. of elements in β in exactly one way.

Pf: It is a combination of things we already know.

10.10.06 ~~Lecture~~

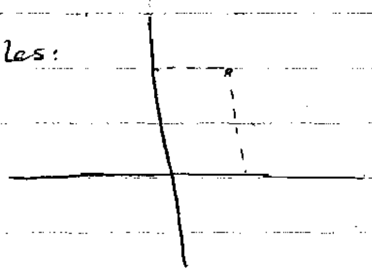
Thm: If a vector space V has a finite basis then every other basis of V is also finite and has same number of elements

Corollary: If V has a finite basis, it makes sense to define

$\dim V$ = the number of elements in a basis of V
" the dim of V

In this case, we say that V is "finite dimentional"

Examples:



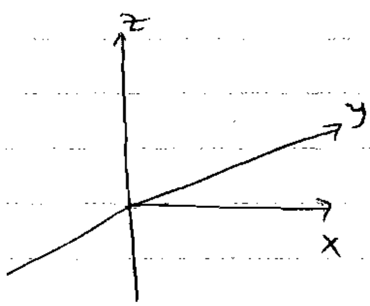
$= \mathbb{R}^2$

basis : $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$\left\{ \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} -\pi \\ e \end{pmatrix} \right\}$

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\dim \mathbb{R}^2 = 2$



$= \mathbb{R}^3$

basis = $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\mathbb{R}^4 = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \right\}$ basis : $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \cdots \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\dim \mathbb{R}^4 = 4$

$\dim \mathbb{R}^n = n$

From before: If a finite set S generate a v.s. V, then it has a subset $\beta \subset S$ which is a basis.

Prf: If $V = \{0\}$, take $\beta = \emptyset$ and it's a basis.

Otherwise V contains a non-zero vector, hence so does S, so pick $u_1 \in S$ s.t. $u_1 \neq 0$

pick $u_2 \in S$ s.t. $u_2 \notin \text{span}(u_1)$
pick $u_3 \in S$ s.t. $u_3 \notin \text{span}(u_1, u_2)$

Keep going until you cannot find some $u_{k+1} \in S$ s.t. $v_{k+1} \notin \text{span}(u_1, \cdots u_k)$

The process is guaranteed to stop because S is finite. At this stage, every $u \in S$ satisfies $u \in \text{span}(u_1, \cdots u_k)$

So $S \subset \text{span}(u_1, \cdots u_k)$

So $V = \text{span}\, S \subset \text{span}(u_1, \cdots u_k)$

So if $\beta = \{u_1, \cdots u_k\}$ then β generates V

<u>Claim:</u> β is lin. indep., hence it is a basis.

Indeed, assume

$$a_1u_1 + a_2u_2 + \cdots + a_ku_k = 0$$

Furthermore, assume j is the maximal index for which $a_j \neq 0$

(if all $a_j = 0$, we are done!)

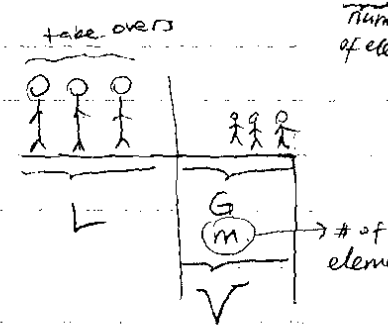
So

$$a_1u_1 + a_2u_2 + \cdots + a_ju_j = 0 \quad / \div a_j$$

$$\frac{a_1}{a_j}u_1 + \frac{a_2}{a_j}u_2 + \cdots + \frac{a_{j-1}}{a_j}u_j + u_j = 0$$

$$\Rightarrow u_j = -\frac{a_1}{a_j}u_1 - \cdots - \frac{a_{j-1}}{a_j}u_j \qquad \text{So } u_j \in \operatorname{Span} u_1 \ldots u_{j-1} \;\lightning$$

<u>Lemma:</u> Suppose a finite set G generates a v.s. V and assume $\underbrace{|G|}_{\text{Numb of elem}}$ -



$P(n)$ Suppose L is a lin. indep. subset of V and $|L| = n$. Then $n \leq m$ and there exists a subset $H \subset G$ s.t. $|H| = m - n$ and span $H \cup L = V$ ("The replacement Lemma")

Prf of the main thm; assuming the lemma:

Let α and β be basis of V, and assume α is finite and $|\alpha| = m$.

Take $G = \alpha$ and for some n, take $L = \{\beta_1, \ldots, \beta_n\}$ the first n elements of β. Then G generates as α is a basis, L is lin. indep. as $L \subset \beta$ & β is lin. indep., so by the lemma, $n \leq m$.

So β must be finite and $|\beta| \leq m$.

So $|\beta| \leq |\alpha|$. But now that we know that both α & β are finite, rerun the argument with α & β switching roles, and therefore

$|\alpha| \leq |\beta|$ ∎

$$\underbrace{\overset{G}{\{u_1, u_2, \cancel{x_3}, \ldots, u_m\}} \quad \overset{L}{\{v_1, v_2, v_3, \ldots, v_n\}}}_{V \ni v_4}$$

u_2 has a non-zero coeff.

u_2 is a linear comb. of $u_1, \ldots, u_m, v_1, \ldots, v_4$

<u>Proof</u> by induction on n, with $P(n)$ as below

Then $a_{m-n}^{-1} v_{n+1} = \frac{a_1}{a_{m-n}} u_1 + \cdots + \frac{a_{m-n-1}}{a_{m-n}} u_{m-n-1} + 1 \cdot u_{m-n} + \frac{b_1}{a_{m-n}} v_1 + \cdots + \frac{b_n}{a_{m-n}}$

So

$$u_{m-n} = c_1 u_1 + c_2 u_2 + \cdots + c_{m-n-1} u_{m-n-1} + d_1 v_1 + \cdots + d_{n+1} v_{n+1}$$

So $u_{m-n} \in \text{span}\{u_1, \ldots, u_{m-n-1}\} \cup \{v_1, \ldots, v_{n+1}\}$

So $\overset{V}{\overset{\shortparallel}{\text{span}\{u_1, \ldots, u_{m-n}, v_1, \ldots, v_n\}}} \subset \text{span}\{u_1 \ldots u_{m-n-1}\} \cup \{v_1 \ldots v_{n+1}\}$

So $V = \text{span}\{\underbrace{u_1 \ldots u_{m-n-1}}_{H}\} \cup \{v_1 \ldots v_{n+1}\}$ ∎

$P(n)$ $m \geq n$

$\& \ \exists H \ldots |H| = m-n, \ldots$

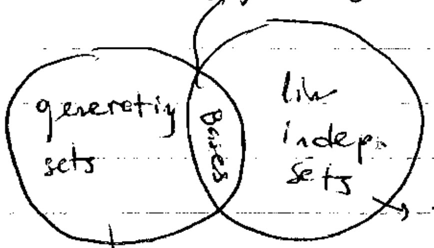
$P(n+1)$: $m \geq n+1$

$\& \ \exists H \ |H| = m-(n+1) = m-n-1$ ✓

The more fact fact we found H with $|H| = m-n-1$ implies $m-n-1 \geq 0$

$\Rightarrow m \geq n+1$

just right



too small or just right

too big or just right

Thm: Let V be a v.s. and assume $\dim V = n$

Then 1. If span $G = V$ then $|G| \geq n$ and if $|G| = n$, then G is a basis,

2. If L is linearly indep., then $|L| \leq n$ & if $|L| = n$, then L is a ba

Proof: (part 1)

If span $G = V$ then there is $\beta \subset G$ which is a basis, so $|\beta| = n$ so

$|G| \geq |\beta|$ so $|G| \geq n$

If $|G| = n$, then $|G| = |\beta| \Rightarrow G = \beta$ so G is a basis.

Pf of 2:

Assume L is lin. indep., and take a basis β of V so $|\beta| = n$.

Take $G = \beta$ and apply the lemma: then $|L| \leq |G| = |\beta| = n$

Furthermore, $\exists$ $H \subset G$ s.t. $|H| = |G| - |L|$ and if $|L| = n$ then $|H| = |G| - |$

$= n - n = 0$ so $H = \emptyset$

and so that span $L \cup H$ = span G = span $\beta = V$

If $|L| = n$, then $H = \emptyset$, $L \cup \emptyset = L$ so span $L = V$

so L is generating $\Rightarrow$ L is a basis.