Lecture Plans

26.10.06 ① All you can say about lin transformations without fixing a basis.

② Then choose a basis.

Recalls $T: V \to W$ is a "lin. trans." if

1. $T(0) = 0$
2. $T(x+y) = T(x) + T(y)$ $\overset{\text{claim}}{\Leftrightarrow}$ $T(cx+y)$
3. $T(cx) = cT(x)$ $\Rightarrow \forall$ $c\,T(x) + T(y)$

$x, y \in V \quad c \in F$

$$T(cx+y) \overset{2}{=} T(cx) + T(y) \overset{3}{=} c\,T(x) + T(y)$$

Examples:

① $O: V \to W \qquad x \in V \quad O(x) = O_W$

② (1) $Id_V = I_V$ "The identity transformation of V"

$$I_V : V \to V \quad I_V(x) = x$$

$$I_V(cx+y) \overset{?}{=} c\,I_V(x) + I_V(y)$$

$$cx+y \overset{!}{=} cx+y$$

Given any linear transformation $T: V \to W$

Definition:

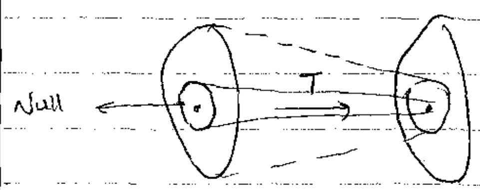
$$N(T) = \operatorname{Ker} T = \begin{matrix}\text{Null space of } T \\ \text{Kernel of } T\end{matrix} := \{x \in V \mid T(x) = \mathbf{0}_W\}$$

$N(0) = V \qquad N(I_{dv}) = \{0_v\}$

② $\begin{array}{ccc} T: V & \longrightarrow & W \\ \cup & & \cup \\ N(T) & & R(T) \end{array}$

In the text

$R(T) = I_m(T) =$ range of T / image of T $= \left\{ w \in W \;\middle|\; \begin{array}{l} \exists\, x \in V \\ \text{s.t. } w = Tx \end{array} \right\} = \left\{ Tx \mid x \in V \right\}$

N · 

$R(0) = \{0_w\} \qquad R(I_{dv}) = V$

<u>Proposition</u>: ① $N(T)$ is a subspace of V

② $R(T)$ is a subspace of W

$\frac{1}{2}$ <u>Proof</u> ① $x, y \in N(T)$

$T(x) = 0 = T(y)$

$\Downarrow$

$T(x+y) = T(x) + T(y) = 0 + 0 = 0$

$\Rightarrow x + y \in N(T)$

$W_1 = T(x_1)$
$W_2 = T(x_2)$

$\frac{1}{2}$ Prf ② $w_1, w_2 \in R(T)$ (It means $\exists\, x_1 \in V, x_2 \in V$ s.t $w_i = T(x_i)$)

So $T(x_1 + x_2) = T(x_1) + T(x_2) = w_1 + w_2$

hence $w_1 + w_2 \in R(T)$

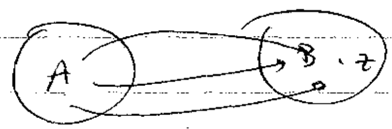
Definition:

① $\dim N(T) =: nullity(T)$

② $\dim R(T) =: rank(T)$

Proposition:

Feedback: $F : A \to B$



① F is 1-1 if $f(x) = f(y) \Rightarrow x = y$ $f(g(z)) = z$

② F is "onto" and 1-1 $\Leftrightarrow$ f is "invertible" $\exists\, g : B \to A$ s.t $g(f(x)) = x$

Prop. ① $Nullity(T) = 0 \Leftrightarrow T$ is 1-1

② $Rank(T) = \dim W \Leftrightarrow T$ is onto

Prf2 $R(T) \subset W$

$rank T = \dim R(T) = \dim(W) \Leftrightarrow R(T) = W \Leftrightarrow T$ is onto.

Prf 1 $\Leftarrow$ assume T is 1-1

Let $x \in N(T)$, then $T(x) = 0 = T(0)$

so $T(x) = T(0)$, by 1-1 $\Rightarrow x = 0$

$\Rightarrow N(T) = \{0\} \Rightarrow \text{nullity}(T) = 0$

$\Rightarrow$ assume $\text{nullity}(T) = 0 \Rightarrow N(T) = \{0\}$

assume $T(x) = T(y) \Rightarrow T(x) - T(y) = 0$

$T(x-y) = 0$

$\Rightarrow x-y \in N(T)$

$\Rightarrow x-y = 0$

$\Rightarrow x = y$ ∎

Thm (The dimension theorem or rank-nullity thm)

only depends on the Domain (It doesn't depend on the transformations)

$\text{rank}(T) + \text{nullity}(T) = \dim V$

Examples:

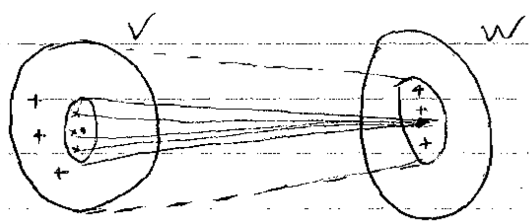
$D_1 : P_n(\mathbb{R}) \longrightarrow P_{n-1}(\mathbb{R}) \qquad D_1(P) = P'$

$D_0 = P_n(\mathbb{R}) \longrightarrow P_n(\mathbb{R}) \qquad D_0(P) = P'$

T	$N(T)$	nullity (T)	$R(T)$	rank(T)	dim V
$D_{\to}$ onto	{constants}	1	$P_{n-1}(\mathbb{R})$ everything	n	$n+1$
D_0 not onto	{constants}	1	$P_{n-1}(\mathbb{R})$ not everything	n	$n+1$

b/c target is bigger

Pf:



If I want to compute the dim of N then I need to start with bases. Then I extend this bases (non of the basis is 0). These images form a base for R

$N(T) \subset V$ so $N(T)$ is . Pick a basis $\{u_1, \ldots, u_k\}$ of $N(T)$. By an earlier theorem, you can find $\{v_1, \ldots, v_l\} \in V$ s.t. $\{u_1, \ldots, u_k, v_1, \ldots, v_l\}$ is a basis of V.

claim: let $w_i = T(v_i)$. Then $\{w_i\}$ is a basis of $R(T)$

If so, $\dim V = k+l$

$\dim N(T) = k$

$\dim R(T) = l$ and rank-nullity is proven

Proof of Claim: $\{w_i\}$ span

$$R(T) = \{Tv : v \in V\} = \{T(\sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{l} b_i v_i\}$$

$$= \{\sum a_i \underbrace{T(u_i)}_{0} + \sum b_i T(v_i)\}$$

$$= \{\sum b_i w_i\} = \text{span}(w_1, \ldots, w_l)$$

$\{w_i\}$ is lin. indep. Assume some $\sum b_i w_i = 0$

$$T\left(\sum b_i v_i\right) = \sum b_i T(v_i) = \sum b_i w_i = 0$$

$\Rightarrow G = \sum b_i v_i \in N(T)$ therefore, $\exists\, a_i$ s.t. $G = \sum a_i u_i$

$$\sum b_i v_i = G = \sum a_i u_i$$

$$\Rightarrow \sum a_i u_i + \sum (-b_i) v_i = 0$$

but $\{u_i, v_i\}$ is a basis.

$\Rightarrow a_i = 0,\ b_i = 0\ \forall i \Rightarrow b_i = 0\ \forall i.$ $\boxed{}$

Thm: nullity(T) + rank(T) = dim V

Corollary: TFAE (the following are equivalent) when the dim V = dim W

& $T: V \to W$

to say T is 1-1 $\Leftrightarrow$ nullity $=0$

T is onto $\Leftrightarrow$ rank $T = \dim V$

1. T is 1-1

$$1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$$

2. T is onto

$$1 \Leftrightarrow 4$$

3. rank $T = \dim V$

$$2 \Leftrightarrow 3$$

4. nullity $T = 0$

5. T is invertible

$1 \Rightarrow 2$: rank + nullity = dim

rank + 0 = dim

rank = dim ✓ (3) so $1 \Rightarrow 2$ $\quad 2 \Rightarrow 1 \Rightarrow 1 \Leftrightarrow 2$.

$$5 \Leftrightarrow 1,2$$

Thm (challenge) : $T : V \to W$, $T' : V' \to W'$ s.t.

$$(\dim V, \dim W, \operatorname{rank} T) = (\dim V', \dim W', \operatorname{rank} T')$$

then $\exists$ an isomorphisms

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\Phi"'] & W \arrow[d, "\Psi"] \\ V' \arrow[r, "T'"] & W' \end{tikzcd}$$

$\Phi : V \to W$

$\Psi : W \to W'$ s.t.

$$\forall v \in V \quad \Psi(T(v)) = T'(\Phi(v))$$