

Jan 11<sup>th</sup>. Agenda TT: Tue, Jan 17<sup>th</sup> 5-7 PM @ EX300

Extra office hours: Dron Mon Jan 16 5:30-8 BA 6178

Jeff Tue Jan 17 11-2. Huron 215

- Material: Everything from last TT/HW6 until Friday, proportional to time spent + around 20% from older material
- Roughly choose  $\frac{4}{5}$
- About  $\frac{1}{3}$  "prove as in class",  $\frac{1}{3}$  "solve as in hw",  $\frac{1}{3}$  "solve fresh"

Riddle Cars A, B, C, D drive in the Sahara desert on generic straight lines and at constant speed, it is known that A meets B (arrive same place at same time), A meets C, A meets D, B meets C, and B meets D. Does C necessarily meet D?

Def  $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an "isometry" if  $\forall x, y \quad d(hx, hy) = d(x, y)$

Theorem  $h$  is an isometry iff it is of the form  $hx = P + Ax$  where  $A \in M_{n \times n}$ , satisfies  $A^T A = I$

Already Shown WLOG,  $h(0) = 0$ ,  $h$  preserves norms & dot products.

$$A := (h(e_1) \dots h(e_n)) \in O(n) \quad [A^T A = I]$$

Claim  $h(\sum x_i e_i) = \sum x_i h(e_i)$

If true,  $h\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = h(\sum x_i e_i) \stackrel{\text{claim}}{=} \sum x_i h(e_i) = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = Ax$

Pf. Let  $\Delta = h(\sum x_i e_i) - \sum x_i h(e_i)$

$$\langle \Delta, h(e_j) \rangle = \langle h(\sum x_i e_i), h(e_j) \rangle - \langle \sum x_i h(e_i), h(e_j) \rangle$$

$$= \langle h(\sum x_i e_i), h(e_j) \rangle - \sum x_i \langle h(e_i), h(e_j) \rangle$$

$$= \langle \sum x_i e_i, e_j \rangle - \sum x_i \langle e_i, e_j \rangle$$

$$= 0$$

but  $h(e_j) = Ae_j$  so

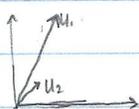
$$0 = \langle \Delta, h(e_j) \rangle = \langle \Delta, Ae_j \rangle$$

$$= \Delta^T Ae_j \quad \forall j \Rightarrow \Delta^T A = 0, \text{ but } A \text{ is invertible, so } \Delta^T = 0 \text{ so } \Delta = 0$$

### Aside (The Gram-Schmidt Process)

If  $\{u_i\}$  is basis of an inner product space [for this class its okay to think  $V = \mathbb{R}^n$ ,  $\langle a, b \rangle = a^T b$ ]. Then there exists (almost unique) orthonormal basis  $\{v_i\}$ , s.t.  $\forall k \quad 1 \leq k \leq n = \dim V$ ,  $\text{span}(u_i)_{i=1}^k = \text{span}(v_i)_{i=1}^k$

Example  $u_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$  &  $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  in  $\mathbb{R}^2$



$$v_1 = \pm \frac{u_1}{\|u_1\|} = \frac{\begin{pmatrix} 3 \\ 4 \end{pmatrix}}{5} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$v_2' = u_2 - \langle u_2, v_1 \rangle v_1$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \left[ (1 \ 1) \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \right] \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$= \begin{pmatrix} 4/25 \\ -3/25 \end{pmatrix}$$

$$v_2 = \frac{v_2'}{\|v_2'\|} = \frac{\begin{pmatrix} 4/25 \\ -3/25 \end{pmatrix}}{1/5} = \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix}$$

Now in general,  $v_1' = u_1$

$$v_1 = \pm \frac{v_1'}{\|v_1'\|}$$

$$v_2' = u_2 - \langle u_2, v_1 \rangle v_1$$

$$v_2 = \pm \frac{v_2'}{\|v_2'\|}$$

$$v_3' = u_3 - \langle u_3, v_1 \rangle v_1 - \langle u_3, v_2 \rangle v_2 \quad v_3 = \pm \frac{v_3'}{\|v_3'\|}$$

$$v_k' = u_k - \sum_{j=1}^{k-1} \langle u_k, v_j \rangle v_j \quad v_k = \pm \frac{v_k'}{\|v_k'\|}$$

Claim. The process works: 1.  $\{v_i\}$  are orthonormal

$$2. \text{Span}(u_i)_{i=1}^k = \text{Span}(v_i)_{i=1}^k$$

pt Exercise / 247

### k-dim Volumes in $\mathbb{R}^n$



Q. Given  $v_1, \dots, v_k$  in  $\mathbb{R}^n$ , what's  $\text{Vol}(\text{spanned by these}) = V(v_1, \dots, v_k)$

Want 1. If  $A^T A = I \quad A \in M_{n \times n}(\mathbb{R})$ ,

$$V(v_1, \dots, v_k) = V(Av_1, \dots, Av_k)$$

2. If  $v_1, \dots, v_k \in \mathbb{R}^k \times \{0_{n-k}\} \subset \mathbb{R}^n$

$$\text{then } v_i = \begin{pmatrix} y_i \\ \vdots \\ 0 \end{pmatrix}_{n-k} \quad \& \quad V(v_1, \dots, v_k) = \det(y_1 \dots y_k)$$

Thm  $V$  exists and is unique.