Homework 12. TUT: R 4-5 PM

Section 23.

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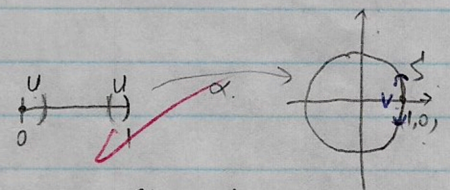
3. a) We can parametrize the set $(x,y) \in S^1$ on the upper semicircle i.e. $y>0$
Set $\phi_1(x) = (x, \sqrt{1-x^2})$ for $x \in (-1,1)$. s.t. ϕ_1 is a smooth map, takes $(-1,1)$ to the upper semicircle. The inverse $\phi_1^{-1}(x,y) = x$ is also smooth. ϕ_1 is diffeomorphism.
For the lower semicircle, take $\phi_2(x) = (x, -\sqrt{1-x^2})$ s.t. $\phi_2(x)$ is a smooth map takes $(-1,1)$ to the lower semicircle. Also, the inverse ϕ_2^{-1} is smooth.
This parametrizes all of S^1 except for two points $(\pm 1, 0)$.
we take $\phi_3 = (\sqrt{1-y^2}, y)$ & $\phi_4 = (-\sqrt{1-y^2}, y)$ to parametrize the two points.
These four parametrizations cover S^1 & for each point at least one of those gives a local parametrization.

b). Consider the point $(1,0)$ in S^1.
Let V be the open nbd. of S^1 containing $(1,0)$.
the function α^{-1} is not continuous. because



there doesnot exist a open set U near point 0 in $[0,1)$ under α. to V. [i.e. the points near $(1,0)$ in S^1 need not map under α^{-1} to points near 0. in $[0,1)$. It will devide into 2 parts, one is near 0 in $[0,1)$ & the other is near 1 in $[0,1)$.]

6. a) Set $\phi_1(x) = x$ on $(0,1)$. S.t. ϕ_1 is an identity function takes points in $(0,1)$ to $(0,1)$ in I.
ϕ_1 is 1-1, onto, & of C^r, α^{-1} is cont. & $D\phi_1(p)$ has rank 1 for each $x \in U$ where $U = V$ is a open set in $H^{(0,1)}$, s.t. $\phi_1 : U \to V$ (a open nbd. of p. $p \in (0,1)$)
Moreover, for point 0, & 1, we take $\phi_2(x) = x$, $\phi_3(x) = 1-x$ where $U = [0, \delta)$ in $\mathbb{R}$.
$\phi_2 : U \to V_2 = [0, \delta)$, $\phi_3 : U \to V_3 = (1-\delta, 1]$ where V_2 & V_3 are open in $[0,1]$. So we parametrize two points.
Therefore, these 2 parametrizations covers $I = [0,1]$ & for each point at least one of these gives a local parametrization.

b). $I \times I$ is NOT a 2-manifold in $\mathbb{R}^2$.
Because when we consider about corners, e.g. $(0,0)$
Assume there exists a coordinate patch $i : [0,1]^2 \to [0,1]^2$.
The differential of i when restricted to $\{0\} \times (0, \varepsilon)$ would be proportional to a vertical tangent vector in the tangent space of $\mathbb{R}^2$, where as the differential when strict to $(0,\varepsilon) \times \{0\}$ would be proportional to a horizontal vector. Since the differential has to be cont. this is a contradiction as the differential at $(0,0)$.

Hilroy

5\. a) let $A \in O(3)$, then $A^T \cdot A = I$, let $A = [x, y, z]$, then $A^T A = \begin{bmatrix} \|x\| & \langle x,y\rangle & \langle x,z\rangle \\ \langle y,x\rangle & \|y\| & \langle y,z\rangle \\ \langle z,x\rangle & \langle z,y\rangle & \|z\| \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So $A = [x, y, z]$ where x, y, z are orthonormal vectors $\in \mathbb{R}^3$, s.t. $\|x\| = \|y\| = \|z\| = I$.

let $f(x,y,z) = (\|x\|^2 - 1, \|y\|^2 - 1, \|z\|^2 - 1, x \cdot y, x \cdot z, y \cdot z)$

So $f(x,y,z) = 0 \Leftrightarrow \|x\|^2 - 1 = \|y\|^2 - 1 = \|z\|^2 - 1 = 0$, $\langle x,y\rangle = \langle x,z\rangle = \langle y,z\rangle = 0$.

$\Leftrightarrow x, y, z$ are orthonormal vectors.

$\Leftrightarrow$ if $A = [x, y, z]$, $A \in O(3)$.

So $f(A) = 0 \Leftrightarrow A \in O(3)$.

b) <u>claim</u>: (Exercise #2 THM): let $f: \mathbb{R}^{n+k} \to \mathbb{R}^n$ be of class C^r. Let M be the set of all x s.t. $f(x) = 0$. Assume that M is non-empty & that $Df(x)$ has rank n for $x \in M$.

Then M is a k-manifold without boundary in $\mathbb{R}^{n+k}$.

<u>proof</u>: define $g: \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ s.t. $g(x_1, \dots, x_{n+k}) = (x_1, \dots, x_k, f_1(x), \dots, f_n(x))$ has rank $n+k$

then $\operatorname{Det}(Dg(x)) \neq 0$ for $x \in M$. So, by the IFT (inverse function THM), g is a

diffeomorphism between a nbd V containing p & an open nbd U in $\mathbb{R}^{n+k}$.
Moreover, g carries $V \cap M$ in a 1-1 fashion onto the set $U_0 = U \cap \mathbb{R}^k \times \{0\}^n$.
Then we define the projection $\pi: U_0 \to W$ as the projection of U_0 onto its first k coordinates.
This is 1-1 & onto since the last n coordinates of elements in U_0 are 0.
Then let $\alpha = g^{-1} \circ \pi^{-1}$. where α takes W to $V \cap M$.
α is of class C^r because π^{-1} is C^∞ & g^{-1} is C^r (since g is C^r)
α^{-1} is cont since g & π are cont.
$D\alpha(x)$ has rank k for $x \in M$. Since the first k rows of $D\alpha$ are independent.
So. M is a k-manifold without boundary in $\mathbb{R}^{n+k}$.
Now back to the question & consider $Df(A)$ where $A \in O(3)$, $A = [x\ y\ z]$

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & 0 & 0 \\ 0 & \frac{\partial f_2}{\partial y} & 0 \\ 0 & 0 & \frac{\partial f_3}{\partial z} \\ \frac{\partial \langle x,y\rangle}{\partial x} & \frac{\partial \langle x,y\rangle}{\partial y} & 0 \\ \frac{\partial \langle x,z\rangle}{\partial x} & 0 & \frac{\partial \langle x,z\rangle}{\partial z} \\ 0 & \frac{\partial \langle y,z\rangle}{\partial y} & \frac{\partial \langle y,z\rangle}{\partial z} \end{bmatrix}$$

Since $\frac{\partial f_1}{\partial x} = \frac{\partial(\|x\|^2-1)}{\partial x} = [2x_1\ 2x_2\ 2x_3] = 2x$ where $x = (x_1, x_2, x_3)$
because $\|x\|^2 - 1 = x_1^2 + x_2^2 + x_3^2 - 1$
$\frac{\partial\langle x,y\rangle}{\partial x} = [y_1\ y_2\ y_3] = y$. where $y = (y_1, y_2, y_3)$
because $\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3$
Similarly, we can get other partial derivatives.

So $Df(x,y,z) \begin{bmatrix} 2x & 0 & 0 \\ 0 & 2y & 0 \\ 0 & 0 & 2z \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{bmatrix}$.

Hence, the rows of $Df(A)$ are independent.
& since $f(A) = 0$, for $A \in O(3)$, not empty by the claim, we get
$O(3)$ is a compact 3-manifold in $\mathbb{R}^9$ without boundary.