Problem Set 11 — MAT257

January 25, 2017

Problems marked with $*$ are to be sumbitted for credit.

1 Munkres §17 (p.151)

$*$ 6. Let S be the tetrahedron in $\mathbb{R}^3$ having vertices $(0,0,0)$, $(1,2,3)$, $(0,1,2)$, and $(-1,1,1)$. Let $f : S \to \mathbb{R}$ be given by

$$f(x,y,z) = x + 2y - z.$$

Evaluate $\int_S f$.

Hint: Use a suitable linear transformation g as a change of variables.

7. Let $0 < a < b$. If one takes the circle in the xz-plane of radius a centred at the point $(b,0,0)$, and if one rotates it about the z-axis, one obtains a surface called the **torus**. If one rotates the corresponding circular disc instead of the circle, one obtains a 3-dimensional solid called the **solid torus**. Find the volume of this solid torus.

Hint: One can proceed directly, but it is easier to use the **cylindrical coordinate transformation**

$$g(r,\theta,z) = (r\cos\theta, r\sin\theta, z).$$

The solid torus is the image under g of the set of all (r,θ,z) for which

$$(r-b)^2 + x^2 \le a^2,$$
$$0 \le \theta \le 2\pi.$$

2 Munkres §20 (p.167)

$*$ 2. Show that if h is an orthogonal transformation, then h carries every orthonormal set to an orthonormal set.

3 Munkres §21 (p.187)

$*$ 3. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be the function $h(\mathbf{x}) = \lambda\mathbf{x}$. If $\mathcal{P}$ is a k-dimensional parallelopiped in $\mathbb{R}^n$, find the volume of $h(\mathcal{P})$ in terms of the volume of $\mathcal{P}$.

5. Prove the following:
Theorem. Let W be an n-dimensional vector space with an inner product. Then there exists a uniquie real-valued function $V(\mathbf{x_1},\dots,\mathbf{x_k})$ of k-tuples of vectors of W such that:

(i) Exchanging $\mathbf{x_i}$ with $\mathbf{x_j}$ does not change the value of V.

(ii) Replacing $\mathbf{x_i}$ by $\mathbf{x_i} + c\mathbf{x_j}$ (for $j \ne i$) does not change the value of V.

(iii) Replacing $\mathbf{x_i}$ by $\lambda\mathbf{x_i}$ multiplies the value of V by $|\lambda|$.

(iv) If the $\mathbf{x_i}$ are orthonormal, then $V(\mathbf{x_1}, \ldots, \mathbf{x_k}) = 1$.

Proof:

(a) Prove uniqueness. [*Hint:* Use the Gram-Schmidt process.]

(b) Prove existence. [*Hint:* If $f : W \to \mathbb{R}^n$ is a linear transformation that carries an orthonormal basis to an orthonormal basis, then f carries the inner product on W to the dot product on $\mathbb{R}^n$.]

4 Munkres §22 (p.193)

1. Let A be open in $\mathbb{R}^k$; let $\alpha : A \to \mathbb{R}^n$ be of class $\mathcal{C}^r$; let $Y = \alpha(A)$. Suppose $h : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry; let $Z = h(Y)$ and let $\beta = h \circ \alpha$. Show that Y_α and Z_β have the same volume.

* 2. Let A be open in $\mathbb{R}^k$; let $f : A \to \mathbb{R}$ be of class $\mathcal{C}^r$; let Y be the graph of f in $\mathbb{R}^{k+1}$, parametrized by the function $\alpha : A \to \mathbb{R}^{k+1}$ given by $\alpha(\mathbf{x}) = \big(\mathbf{x}, f(\mathbf{x})\big)$. Express $v(Y_\alpha)$ as an integral.

Hint: In some approaches, it may be possible to simplify the end result using the formula

$$\det(I_{n\times n} + vw^T) = 1 + v^T w$$

which holds for vectors $v, w \in \mathbb{R}^n$. — **DBN**

5 "In addition…"

A. **Curves!**
Consider a simple curve given by a smooth map $\gamma : [0, T] \to \mathbb{R}^n$, and let $\mathcal{C} := \operatorname{Im}(\gamma)$.

1. Prove that the arclength $\lambda(\mathcal{C})$ of the curve $\mathcal{C}$ is given by

$$\lambda(\mathcal{C}) = \int_0^T \|\dot\gamma(t)\|\, dt$$

where $\|\cdot\|$ is the Euclidian norm on $\mathbb{R}^n$ and $\dot\gamma = \frac{d}{dt}\gamma$.

2. Consider the length at time $t \in [0, T]$ of γ,

$$s(t) = \int_0^t \|\dot\gamma(u)\|\, du.$$

Let $\hat\gamma : [0, \lambda(\mathcal{C})] \to \mathbb{R}^n$ be the reparametrization of $\mathcal{C}$ in terms of arclength (i.e. $\mathcal{C} = \operatorname{Im}(\hat\gamma)$). Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous in a neighbourhood of $\mathcal{C}$; express the integral f in terms of the parametrizations γ and $\hat\gamma$.

3. Consider the helix $\mathcal{C} \subset \mathbb{R}^3$ parametrized as follows:

$$\gamma(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ 2t \end{pmatrix}, \quad t \in [-2\pi, 2\pi].$$

Compute the arclength $\lambda(\mathcal{C})$ and the integral $\int_{\mathcal{C}} f$ where $f(x, y, z) = xy\sin z$.

B. **Surfaces!**
A parametrization of an orientable surface $\mathcal{S} \subset \mathbb{R}^3$ is a $\mathcal{C}^r$ map

$$\begin{aligned} \sigma :\ & \mathcal{U} \to \mathbb{R}^3 \\ & (u, v) \mapsto (x(u, v),\ y(u, v),\ z(u, v)) \end{aligned}$$

where $\mathcal{U} \subset \mathbb{R}^2$ is open.

∗ 1. Compute the area $\mathcal{A}(\mathcal{S})$ of the surface $\mathcal{S} \subset \mathbb{R}^3$ given by the parametrization

$$\sigma(u, v) = (u \cos v,\ u \sin v,\ v)$$

$\forall\ (u, v) \in (0, 1) \times (0, \pi)$.

2. Let $\mathcal{S}$ be the part of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = b^2$ where $0 < b < a$. Compute the surface area $\mathcal{A}(\mathcal{S})$.