October 29th

<u>MAT 240 – Linear Algebra Lecture</u>

Reminder: Choosing a basis, V is isomorphic to F^n

Goal:

1. The set $L(\mathrm{V}, \mathrm{W})$ of all linear transformations $V \to W$ is a V.S.
2. Choosing bases, it is isomorphic to $M_{mxn}(F)$ $m = \dim(V)$ $n = \dim(W)$

Extra Claim (from last class final example):

If two linear transformations $S, S^{`}: X \to Y$ agree on a basis of X, they are equal.

If (x_i) is a basis of X and $\forall i\ S(x_i) = S^{`}(x_i) \in Y$ then $S = S^{`}$

Proof: Pick some element $x \in X$, as (x_i) as a basis, find scalars a_i such that.

$$x = \sum a_i x_i$$

Now: $S(x) = S(\sum a_i x_i) = \sum a_i S(x_i) = \sum a_i S^{`}(x_i) = \sum a_i\, S^{`}(x_i) = S^{`}(\sum a_i x_i) = S^{`}(x)$

$\to S = S^{`}$ ■

Let $\beta = (u_{1,} \cdots u_n)$ be a basis (**basis is ordered**) of a finite dimension vector space V. $x \in V$ $x = \sum a_i u_i$

Let $[x]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = T(x)$

$\beta = (u_{1,} \cdots u_n)$ of V

$\gamma = (e_1 \cdots e_n)$ of F^n

$T: V \to F^n$

Defined by:

$u_i \to e_i$

Indeed, $T(x) = T(\sum a_i u_i) = \sum a_i T(u_i) = \sum a_i e_i = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = [x]_\beta$

$T: V \to W$ is a linear transformation, $\beta = (v_{1,} \cdots v_n)$ is a basis for V, $\gamma = (w_{1,} \cdots w_n)$ is a basis for W

$$A = [T]_\beta^\gamma = \begin{bmatrix} [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \end{bmatrix}$$

In $P_2(\boldsymbol{R})$:

$$[x^2 - 2x + 3]_{(x^2,x,1)} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$[x^2 - 2x + 3]_{(1,x,x^2)} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$[x^2 - 2x + 3]_{(x^2,x,3)} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Coordinates depend on choice of basis!

D: $P_3(\boldsymbol{R}) \to P_2(\boldsymbol{R})$ (Differentiation)

$$\beta = (x^3, x^2, x, 1) \qquad \gamma = (x^2, x, 1)$$

$$[D]_\beta^\gamma = [D]_{(x^3,x^2,x,1)}^{(x^2,x,1)} = \begin{pmatrix} [D(x^3)]_\gamma & [D(x^2)]_\gamma & [D(x)]_\gamma & [D(1)]_\gamma \end{pmatrix}$$

$$= \begin{pmatrix} [3x^2]_\gamma & [2x]_\gamma & [1]_\gamma & [0]_\gamma \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$T_\theta: \boldsymbol{R}_2 \to \boldsymbol{R}_2$

$$[T_\theta]_{e_1,e_2}^{e_1,e_2} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$