Problem Set 9 — MAT257

November 30, 2016

Problems marked with $*$ are to be sumbitted for credit.

1 Munkres §13 (p.111)

1. Let $f, g : S \to \mathbb{R}$. Assume f and g are integrable over S.
   (a) Show that if f and g agree except on a set of measure zero, then $\int_S f = \int_S g$.
   (b) Show that if $f(\mathbf{x}) \leq g(\mathbf{x})$ for $\mathbf{x} \in S$ and $\int_S f = \int_S g$, then f and g agree except on a set of measure zero.
2. Let A be a rectangle in $\mathbb{R}^k$; let B be a rectangle in $\mathbb{R}^n$; let $Q = A \times B$. Let $f : Q \to \mathbb{R}$ be a bounded function. Show that if $\int_Q f$ exists, then
   $$\int_{\mathbf{y}\in B} f(\mathbf{x}, \mathbf{y})$$
   exists for $\mathbf{x} \in A \setminus D$, where D is a set of measure zero in $\mathbb{R}^k$.
3. Complete the proof of **Corollary 13.4**.
4. Let S_1 and S_2 be bounded sets in $\mathbb{R}^n$; let $f : S_1 \cap S_2 \to \mathbb{R}$ be a bounded function. Show that if f is integrable over S_1 and S_2, then f is integrable over $S1 \setminus S_2$, and
   $$\int_{S1\setminus S_2} f = \int_{S_1} f - \int_{S1\cap S_2} f.$$
5. Let S be a bounded set in $\mathbb{R}^n$; let $f : S \to \mathbb{R}$ be a bounded continuous function; let $A = \mathrm{Int} S$. Give an example where $\int_A f$ exists and $\int_S f$ does not.
6. Show that **Theorem 13.6** holds without the hypothesis that f is continuous on S.
7. Prove the following:
   Theorem. Let S be a bounded set in $\mathbb{R}^n$; let $f : S \to \mathbb{R}$ be a bounded function. Let D be the set of points of S at which f fails to be continuous. Let E be the set of points of $\mathrm{Bd}S$ at which the condition
   $$\lim_{\mathbf{x}\to\mathbf{x_0}} f(\mathbf{x}) = 0$$
   fails to hold. Then $\int_S f$ exists if and only if D and E have measure zero.
   Proof.
   (a) Show that f_S is continuous at each point $\mathbf{x_0} \notin D \cup E$.
   (b) Let B be the set of isolated points of S; then $B \subset E$ because the limit cannot be defined if $\mathbf{x_0}$ is not a limit point of S. Show that if f_S is continuous at $\mathbf{x_0}$, then $\mathbf{x_0} \notin D \cup (E \setminus B)$.
   (c) Show that B is countable.
   (d) Complete the proof.

2 Munkres §14 (pp.120-121)

1. Let S be a bounded set in $\mathbb{R}^n$ that is the union of the countable collection of rectifiable sets $S_1, S_2, \ldots$.
   (a) Show that $S_1 \cup \cdots \cup S_n$ is rectifiable.
   (b) Give an example showing that S need not be rectifiable.
2. Show that if S_1 and S_2 are rectifiable, so is $S_1 \setminus S_2$, and
$$v(S_1 \setminus S_2) = v(s_1) - v(S_1 \cap S_2).$$
3. Show that if A is a nonempty, rectifiable open set in $\mathbb{R}^n$, then $v(A) > 0$.

$*$ 4. Give an example of a bounded set of measure zero that is rectifiable, and an example of a bounded set of measure zero that is not rectifiable.

5. Find a bounded closed set in $\mathbb{R}$ that is not rectifiable.
6. Let A be a bounded open set in $\mathbb{R}^n$; let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded continuous function. Give an example of where $\int_{\overline{A}} f$ exists but $\int_A f$ does not.
7. Let S be a bounded set in $\mathbb{R}^n$.
   (a) Show that if S is rectifiable, then so is the set $\overline{S}$, and $v(S) = v(\overline{S})$.
   (b) Give an example where $\overline{S}$ and Int$S$ are rectifiable, but S is not.
8. Let A and B be rectangles in $\mathbb{R}^k$ and $\mathbb{R}^n$, respectively. Let S be a set contained in $A \times B$. For each $\mathbf{y} \in B$, let
$$S_\mathbf{y} = \{\mathbf{x} : \ \mathbf{x} \in A \ \text{ and } \ (\mathbf{x}, \mathbf{y}) \in S\}.$$
   We call $S_\mathbf{y}$ a **cross-section** of S. Show that if S is rectifiable, and if $S_\mathbf{y}$ is rectifiable for each $\mathbf{y} \in B$, then
$$v(S) = \int_{\mathbf{y} \in B} v(S_\mathbf{y}).$$

3 Munkres §15 (pp.132-133)

$*$ 4. Let $f(x, y) = 1/(y + 1)^2$. Let A and B be the open sets
$$A = \{(x, y) : \ x > 0 \ \text{ and } \ x < y < 2x),$$
$$B = \{(x, y) : \ x > 0 \ \text{ and } \ x^2 < y < 2x^2),$$
of $\mathbb{R}^2$. Show that $\int_A f$ does not exist. Show that $\int_B f$ *does* exist and compute it.

$*$ 8. Let A be open in $\mathbb{R}^n$. We say $f : A \to \mathbb{R}$ is **locally bounded** on A if each $\mathbf{x} \in A$ has a neighbourhood on which f is bounded. Let $\mathcal{F}(A)$ be the set of all functions $f : A \to \mathbb{R}$ that are locally bounded on A and continuous on A except on a set of measure zero.
   (a) Show that if f is continuous on A, then $f \in \mathcal{F}(A)$.
   (b) Show that if f is in $\mathcal{F}(A)$, then f is bounded on each compact subset of A and the definition of the extended integral $\int_A f$ goes through without change.
   (c) Show that Theorem 15.3 holds for functions f in $\mathcal{F}(A)$.
   (d) Show that Theorem 15.4 holds if the word “continuous” in the hypothesis is replaced by “continuous except on a set of measure zero.”

4 "In addition..."

∗ A. Compute the volume of the "2D ice cream cone".

$$C = \{(x,y) : \ |x| \leq y \leq 1 + \sqrt{1-x^2}\}.$$

∗ B. Compute the volume of the "n-dimensional simplex".