26.

(a) Proof:

① For $\forall x, y \in V$, $c \in F$, $x = x_1 + x_2$, $y = y_1 + y_2$, $x_1, y_1 \in W_1$, $x_2, y_2 \in W_2$.

$T(cx+y) = T(c(x_1+x_2)+(y_1+y_2))$

$= T((cx_1+y_1)+(cx_2+y_2))$

$\because x_1, y_1 \in W_1$, $x_2, y_2 \in W_2$, W_1, W_2 are all subspaces

$\therefore T(cx+y) = T((cx_1+y_1)+(cx_2+y_2))$

$= cx_1 + y_1$

$cT(x) + T(y) = cT(x_1+x_2) + T(y_1+y_2)$

$= cx_1 + y_1$

Hence $T(cx+y) = cT(x) + T(y)$ i.e. T is linear

② $\because$ For $\forall x_1 \in W_1$ & $0 \in W_2$ then we have

$x = x_1 + 0$, $x \in V$ from definition.

and $T(x) = T(x_1+0) = x_1 = x$.

Hence for all x satisfy $T(x) = x$, $x \in V$ and $x \in W_1$

Then $W_1 = \{x \in V : T(x) = x\}$

(b) Proof: Suppose β is a basis of V.

$\because V = W_1 \oplus W_2 \therefore$ for β_1 which is a basis of W_1

β_2 which is a basis of W_2

$\beta = \beta_1 \oplus \beta_2$ from problem 29 in Ex. 1.6.

Suppose $\beta_1 = \{x_1, x_2, \cdots, x_m\}$ $\beta_2 = \{y_1, y_2, \cdots, y_n\}$

$R(T) = \text{Span}(T(\beta)) = \text{Span}(T(\beta_1), T(\beta_2))$

$= \text{Span}(T(x_1), T(x_2), \cdots, T(x_m), T(y_1) \cdots T(y_n))$

$= \text{Span}(x_1, x_2, \cdots, x_m, T(0+y_1) \cdots T(0+y_n))$

$= \text{Span}(x_1, x_2, \cdots x_m, 0, \cdots, 0)$

$= \text{Span}(x_1, x_2, \cdots, x_m)$

$= W_1$

$N(T) = \{x \in V : T(x) = 0\}$

Suppose $x = x_1 + x_2$, $x_1 \in W_1$, $x_2 \in W_2$

$\Rightarrow T(x) = x_1 = 0 \Rightarrow x_1 = 0. \Rightarrow x = x_2$

$\Rightarrow \forall x \in V,\ x \in W_2$

$\Rightarrow N(T) = W_2$

(c) if $W_1 = V$

$\Rightarrow \forall x \in V,\ x \in W_1$

$\Rightarrow T(x) = x_1 = x$

$\Rightarrow T: V \to V,\ \forall x \in V,\ T(x) = x$

(d) if $W_1 = \{0\}$

$\Rightarrow \forall x \in V$, we suppose $x = 0 + x_2,\ x_2 \in W_2$

$\Rightarrow T(x) = T(0 + x_2) = 0$

$\Rightarrow T: V \to V,\ \forall x \in V,\ T(x) = 0$

2) (a) Proof:

Suppose that $\beta = \{u_1, u_2, \cdots u_m, v_1, v_2, \cdots, v_n\}$ is a basis of V and $\beta_1 = \{u_1, u_2, \cdots, u_m\}$ is a basis of W. Then we create $W' = \text{span}\{v_1, v_2, \cdots, v_n\}$, then by β is linearly independent, $\beta_2 = \{v_1, v_2, \cdots v_n\}$ is a basis of W'.

We now want to show that $W \oplus W' = V$

① To show $W + W' = V$

$\because \beta = \beta_1 \cup \beta_2$, $\text{span}(\beta) = V$

$\therefore \text{span}(\beta_1 \cup \beta_2) = V \quad \therefore W + W' \supseteq V$.

$\because \beta_2 \subseteq \beta \quad \therefore W' = \text{span}(\beta_2) \subseteq \text{span}(\beta) = V$

$\because W \subseteq V,\ W' \subseteq V$, V is a vector space

$\therefore W + W' \subseteq V. \quad \therefore W + W' = V$