Notice that $A(e_{x_\eta}) = \sum_{\nu \in J} e_{x_\eta}(x_\nu) r_{x_\nu} = r_{x_\eta} \in X \subset R^n$.

Define the matrix $[A]$ to be the $n \times |J|$ matrix whose η^{th} column is $A(e_{x_\eta}) = r_{x_\eta} \in X \subset R^n$.

(In the case where J is countable, $J = \{1, 2, \dots\}$, $X = \{x_j : j \in J\}$,
$A(e_{x_j}) = \sum_{l=1}^{\infty} e_{x_j}(x_l) r_{x_l} = r_{x_j} \in X \subset R^n$, so $[A] = \left(A(e_{x_1}), A(e_{x_2}), \dots, A(e_{x_j}), \dots \right)$.)

Notice that for arbitrary fcn $a \in R^X$, we can write a as $|J| \times 1$ column vector wrt the "basis" $\{e_\eta : \eta \in J\}$ of R^X, namely $a = \sum_{\eta \in J} a(x_\eta) e_{x_\eta}$ $\left(= \sum_{x_\nu \in \operatorname{supp}(a)} a(x_\nu) e_{x_\nu} \right)$ So it's a finite sum ☺

$a(x_\eta)$: coordinates in R; e_{x_η}: "basis vectors for R^X"

Recall: $A(e_{x_\eta}) = r_{x_\eta}$. So, $A(a) = A\left(\sum_{\eta \in J} a(x_\eta) e_{x_\eta} \right) = \sum_{\eta \in J} a(x_\eta) A(e_{x_\eta})$ (Since A is an R-mod morph)

Then, by extending the def^n of matrix mult, we have that $A(a) = [A] \left(\begin{matrix} \text{the } |J| \times 1 \text{ column} \\ \text{vector whose} \\ \eta^{th} \text{ entry is } a(x_\eta) \end{matrix} \right)$

In the case where $J = \{1, 2, \dots\}$,

$$A(a) = [A] \begin{pmatrix} a(x_1) \\ a(x_2) \\ \vdots \end{pmatrix} = \left(A(e_1), A(e_2), \dots \right) \begin{pmatrix} a(x_1) \\ a(x_2) \\ \vdots \end{pmatrix} = \sum_{j=1}^{\infty} a(x_j) A(e_j) \; \left(= A(a) \right)$$

(This is by analogy to linear algebra: If $T: F^k \to F^n$, linear trans.,
$T(x) = T\left(\sum_{j=1}^{k} x_j e_j \right) = \sum_{j=1}^{k} x_j T(e_j)$, so if $[T] = \left(T(e_1), T(e_2), \dots, T(e_k) \right) \in M_{n \times k}(F)$, then
$T(x) = [T] \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$)

So, the R-mod. morphism A can be interpreted as an $n \times |J|$ matrix $[A]$ whose η^{th} column is $A(e_\eta)$.
(acting by multiplication on $|J| \times 1$ column vectors in R of which only finitely many entries are nonzero)

!!!

Just like in linear algebra, we only need to know where the $\underbrace{\text{"basis" vectors}}_{e_{x_\eta}\text{'s}}$ of R^X are mapped by A.

where elements of R^X as "coordinate vectors" wrt the e_{x_η}'s, i.e. the η^{th} coord. of a is $a(x_\eta)$. $\left(a = \sum_{\eta \in J} a(x_\eta) e_{x_\eta} \right)$