November 26th, 2009

MAT240: Abstract Linear Algebra Lecture:

Determinants:

1. Applications: Mention two, prove one, use none.
2. Formulas: discuss just one.
3. Basic Properties. Our core subject.

Det: $M_{nxn}(F) \to F$

Use 1:

$$det\begin{pmatrix} ---r_1--- \\ \vdots \\ ---r_n--- \end{pmatrix} = vol(paralleolpiped\ spanned\ by\ r_1 \dots r_n)$$

Use 2:

$$\det(A) \neq 0 \quad iff \quad A^{-1}\, \exists$$

Definition:

$$|(a_{11})| := a_{11}$$

$$|A| = \left|\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right| \equiv \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \left|\widetilde{A_{ij}}\right|$$

in general, $\widetilde{A_{ij}}$ is the matrix obtained from A by removing the i^{th} row and j^{th} column.

Eg. $\left|\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = ad - bc$

Eg. $\left|\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right| = (-1)^{1+1} * a_{11}|(a_{22})| = a_{11}a_{22} - a_{12}a_{21}$

Eg. $\left|\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right| = 1 * 4 - 2 * 3 = -2$

Eg. $\left|\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\right| = 1\left|\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}\right| - 2\left|\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}\right| + 3\left|\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}\right| = 1(-3) - 2(-6) + 3(-3) = 0$

→ The above matrix is not invertible ∴ $rank < 3$

Eg. $\left|\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\right| = 0 - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1(-1) + 1(1) = 2$ →this matrix is invertible.

Theorem: The following basic properties hold:

0. $det(I_n) = I$
1. $det\left(E_{ij}^{1} A\right) = -de\,t(A) \rightarrow de\,t\left(E_{ij}^{1}\right) = -1$

   "flipping two rows in A, flips sign of det A."

   Eg. $det\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = 3 * 2 - 4 * 1 = 2$
2. $\det(E_{ic}^{2} A) = c * det(A) \rightarrow det(E_{1c}^{2}) = c$, even if $c = 0$

   "multiplying a row by c multiplies det A by c."

   Eg. $\det\begin{pmatrix} 1 & 2 \\ 21 & 28 \end{pmatrix} = 28 - 42 = -14 = 7(-2)$

   Eg. $\det\begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = 0 * 4 - 0 * 3 = 0$

   Moral: det of a matrix with a row of 0's is 0.
3. $det\left|E_{1jc}^{3} A\right| = detA \rightarrow \det\left(E_{ijc}^{3}\right) = 1$

   "Adding a multiple of one row to another row to another does not change det A".

   E.g $\left|\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}\right| \xrightarrow{r_2 += 2r_1} \left|\begin{pmatrix} 1 & 2 \\ 5 & 8 \end{pmatrix}\right| = -2$

Claim: Using this we can compute det(A) for any A.

Method: Row reduce A to B which is RREF, keeping track of the changes to det.

If B is RREF:

$$detB = \begin{cases} 1\ if\ B = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \\ 0\ if\ B = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \dots & 0 \end{pmatrix} \end{cases}$$