Text in purple = things that Prof. Dror Bar Natan said in class.

## Monday, October 27th

Claim: $M_{n\times n}(R[x]) \cong (M_{n\times n}(R))[x]$.

i.e. "matrices $\bar{w}$ entries as polynomials" = "polynomials $\bar{w}$ coefficients as matrices".

$$\left\{\begin{pmatrix} \sum a_{11_k}x^k & \cdots & \sum a_{1n_k}x^k \\ \vdots & & \vdots \\ \sum a_{n1_k}x^k & \cdots & \sum a_{nn_k}x^k \end{pmatrix}\right\} \qquad \left\{\sum A_k x^k : A_k \in M_{n\times n}(R),\ A_k = (a_{ij_k})\right\}$$

$$\parallel$$

$$\left\{\left(\sum a_{ij_k} x^k\right)\right\}$$

The map is to map coefficients to coefficients.

Caley-Hamilton Theorem

Cayley-Hamilton: "A matrix annihilates its characteristic polynomial"

Let $A \in M_{n\times n}(R)$ $R$ is a commutative ring.

$$R[t] \ni \chi_A(t) := \det(\underbrace{tI - A})$$

$$\parallel$$

$$\sum a_k t^k \qquad \underbrace{\begin{pmatrix} t-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & t-a_{22} & \cdots & -a_{2n} \\ \vdots & & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & t-a_{nn} \end{pmatrix}}_{\in M_{n\times n}(R[t])}$$

$$\det(a_{ij}) = \sum_{\sigma} (-1)^{\sigma} \prod_{i=1}^{n} a_{i\sigma j} \qquad a_{ij} \in S$$

Claim: $\chi_A(A) = 0$. i.e. $\sum a_k A^k = \chi_A(A) = 0$.

Wrong Proof #1:

Diagonalize matrix A, so the entries on the diagonal are the eigenvalues. Since the characteristic polynomials annihilates eigenvalues, it follows.

This is not our proof since we haven't talked about diagonalization, and the ring can be any commutative ring, so we can't diagonalize, and we can't use eigenvalues and eigenvectors.

Wrong Proof #2:

Wrong Proof: $\chi_A(A) = \det(AI - A)$
$= \det(0) = 0.$

You're putting a matrix in a matrix.
The LHS is a matrix and the RHS a scalar so the evaluation makes no sense.
We also didn't use properties of determinant, so this would also be true for the characteristic polynomial defined by trace:

Basically, it's saying that if we could just sub in A into det (tI – A), then we could also sub in A into tr (tI – A), and then the calculation doesn't make sense.

Facts needed for the correct proof:

Definition of Adj A:

Aside: Adj $A$ = "transpose of matrix of minors"

$= \left((-1)^{i+j} \cdot A_{ji}\right)_{ij}$ $\quad A_{ji} = \det\left(A\right)_{ji}$ → removing row j and column i.

Fact about adj A:

$A \cdot \operatorname{adj} A = \operatorname{adj} A \cdot A = \det(A) \cdot I$. over any commutative $R$.

You should have seen this proof in previous courses. The proof of this fact is entirely algebraic, and it doesn't use anything except for addition and multiplication. The entries of A adj A can be reinterpreted as the determinants of the original matrix minus the row of I and column of j and replaced by other things. It's entirely algebra, so it's true over any commutative ring R.

Correct proof:

Main idea of correct proof:

Sub in A into this equation:

$$\chi_A(t)\cdot I = \det(tI - A)I \doteq \left(\sum B_i t^i\right)\cdot\left(tI - At^0\right)$$

Full correct proof:

$$\det(tI-A)\cdot I \overset{\text{in } M_{n\times n}(R[t])}{=} \operatorname{adj}(tI-A)(tI-A) \overset{\text{in } M_{n\times n}(R)[t]}{=} \left(\sum B_i t^i\right)(tI-A) \qquad (*)$$

The second equality there is from the isomorphism $M_{n\times n}(R[x]) \cong (M_{n\times n}(R))[x]$.

Recall that the evaluative map is defined by:

Aside: if $S$ is commutative,

$$ev_u : S[x] \to S.$$
$$\sum a_i x^i \mapsto \sum a_i u^i$$

We would like to use the evaluation map and substitute the matrix A into (*). But the evaluation map is a ring homomorphism only if the A commute with the Bi's. They're matrixes, so even if the ring itself is commutative, we would still have to prove that the matrices commute.

We'll prove this in the lemma (and R doesn't have to be commutative):

Lemma: All the $B_i$'s commute with $A$.

Proof of Lemma: $(tI-A)\operatorname{adj}(tI-A) = \operatorname{adj}(tI-A)(tI-A)$

$$\Rightarrow (tI-A)\left(\sum B_i t^i\right) = \left(\sum B_i t^i\right)(tI-A)$$
$$\Rightarrow A\sum B_i t^i = \left(\sum B_i t^i\right)A.$$
$$\Rightarrow \forall i \ AB_i = B_i A.$$

The first line of the proof is because $A\cdot \operatorname{adj} A = \operatorname{adj} A\cdot A = \det(A)\cdot I$.

Using this lemma, we finish the proof of the Caley Hamilton theorem by evaluating (*) at A:

Hence under $ev_A$

$$\chi_A(\cancel{A}t)\cdot I = \left(\sum B_i t^i\right)(t\cdot I - A t^0)$$

$$\Rightarrow \chi_A(A)\cdot I = \left(\sum B_i A^i\right)(A I - A I)$$

$$= 0.$$

## Monday, November 10th

Direct Sums

2 Definitions: The "set" definition (where addition and scalar multiplication is defined in the obvious way) and the category theory definition using universal property.

Our goal is to prove:

$$M \text{ f.g } / \text{PID } R \Rightarrow M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle \quad p_i \text{ prime } s_i \in \mathbb{Z}_{>0}$$

Main idea of the proof:

Step 1: Show that M is associated with a matrix A. (Roughly speaking, A is associated with the "kernel of M". We will define this specifically.)

Step 2: Show that if we use row operations on the matrix A to get another matrix A', M will also be associated with the matrix A'.

Step 3: Show that we can use Gaussian elimination on A to get to a matrix of this form: $\begin{pmatrix} a_1 & & & & & \\ & a_2 & & & & \\ & & \ddots & & & \\ & & & a_n & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$.

Step 4: Since M is associated with this matrix $\begin{pmatrix} a_1 & & & & & \\ & a_2 & & & & \\ & & \ddots & & & \\ & & & a_n & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$, $M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle$.

Details of the proof:

## Step 1

Defining the obvious map for a finitely generated module, R^n -> M:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \longmapsto \sum a_i g_i \qquad M = \operatorname{im}\pi \cong R^n/\ker\pi.$$
$$R^n \overset{\pi}{\twoheadrightarrow} M \qquad \ker\pi = \langle r_x : x \in X \rangle \to \text{not claiming finite.}$$

Let X be a generating set for ker pi, so that any element in ker pi can be written as rx for some r \in R and x \in X.

Defining another map from X -> R:

$$\{a : X \to R;\ a(x) \neq 0 \text{ for finitely many } x\text{'s}\} = R^X \xrightarrow{A} R^n \overset{\pi}{\twoheadrightarrow} M.$$

Explaining this map in details:

$R^X$ = $\{a : X \to R;\ a(x) \neq 0 \text{ for finitely many } x\text{'s}\}$

We have a map A: $R^X$ -> R^n by defining A(b) = $\sum_{x\in X} b(x)x.$, where b is in R^x. This sum is finite because b(x) \neq 0 for finitely many x's, and $\sum_{x\in X} b(x)x.$ is in R^n because b(x) is in R and x is in ker pi (which is in R^n), so $\sum_{x\in X} b(x)x.$ is a sum of elements in R^n.

$\operatorname{im} A = \ker\pi$ and $M := R^n/\operatorname{im} A.$

Since X is a generating set for ker pi, the image of A is ker pi.

M is isomorphic to R^n/im A:

By the first isomorphism theorem, pi is surjective, so R^n/ker pi = M. But ker pi = im A, so we also know that R^n/im A = M.

A can be interpreted as an $n \times X$ matrix

finite

finite rows, infinitely many columns.

$$R^X = \langle e_x \rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x \\ 0 \\ \vdots \end{pmatrix} x$$

A can be interpreted as an n x X matrix because A maps R^|X| to R^n. An n x X matrix maps something that's |X| dimensional to something that's n dimensional. Furthermore, in each row, there are only finitely many non-zero entries, since anything in R^X only has finitely many non-zero entries (so if we take A(e_x) for each x, we would be summing up only finitely many non-zero entries).

Furthermore, every $n \times X$ matrix $A$ defines a finitely generated module.

The finitely generated module is just the image of the matrix A (i.e., the column space), then projected by the map pi.

Examples: $A = (1) \rightsquigarrow M = R^1/\mathrm{im}\, 1 = \{0\}$.

$A = (a) \rightsquigarrow M = R^1/\mathrm{im}\, a = R/\langle a \rangle$

$A = (0) \rightsquigarrow M = R^1/\mathrm{im}(0) = R/\{0\} = R$.

4 $C = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$ $M_C = M_A \oplus M_B$.

## Thursday November 13

Every f.g. module is $M_A$ for some $A$.

$M$ is f.g. $\Rightarrow \phi : R^n \twoheadrightarrow M$.

$\Rightarrow M \cong R^n/\ker \varphi$

Take $X = \ker \varphi$.

$R^X \to R^n$ : by $e_x \mapsto x$

Last time, we noted that A defines a finitely generated module, and this is the converse. Given a finitely generated module, take X = ker pi (where pi is the obvious projection map). Then define A: R^X -> R^n

by mapping the basis elements of X to itself (since we took the generating set of ker pi X  to be the whole set ker pi, it makes sense).

## Step 2

Claim:
$$\begin{tikzcd} R^X \arrow[r, "A"] & R^n \arrow[d, "P"] \\ R^X \arrow[u, "Q"] \arrow[r, "A'"] & R^n \end{tikzcd}$$

$P \in M_{n\times n}(R)$  $Q \in M_{X\times X}(R)$

If P and Q are invertible, then $M_A = M_{A'}$

We would like to show that if we had such a commutative diagram, then the modules that are generated are equal.

Proof:
$$\begin{tikzcd} R^X \arrow[r, "A"] & R^n \arrow[r] \arrow[d, "P"] & R^n/\operatorname{im} A = M_A \arrow[d, dashed, shift left, "\rho"] \\ R^X \arrow[u, "Q"] \arrow[r, "A'"] & R^n \arrow[r] & R^n/\operatorname{im} A' = M_{A'} \arrow[u, shift left, "\lambda"] \end{tikzcd}$$

$\rho$ defined w/ P and is well defined
$\lambda$ defined w/ $P^{-1}$ and is well defined.

To show that $M_A \cong M_{A'}$.:

Define an isomorphism $\Phi : M_A \to M_{A'}$ by $\Phi([a]_{\operatorname{im} A}) = [P\alpha]_{\operatorname{im} A'}$., where \alpha \in R^n.

To show that this map is well-defined, we show that if $[\alpha]_{\operatorname{im} A} = 0$ then $[P\alpha]_{\operatorname{im} A'} = 0$. If $[\alpha]_{\operatorname{im} A} = 0$, then

$\alpha \in \operatorname{im} A$ so $\alpha = A\beta$ for some $\beta \in R^X$. Let $\gamma = Q^{-1}\beta$, so that

$$P\alpha = PA\beta = PAQQ^{-1}\beta = PAQ\gamma = A'\gamma.$$

, so $[P\alpha]_{\operatorname{im} A'} = 0$.

Now, we would like to put the matrix A into this form A'= $\begin{pmatrix} a_1 & & & & & \\ & a_2 & & & & \\ & & \ddots & & & \\ & & & a_n & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$ by using $A \mapsto A' = PAQ$, where P \in $M_n(R)$ is invertible and $Q \in M_{|X|}(R)$. We can do this by using row/column operations on A, since row operations correspond to invertible matrices P and Q: Permutation

matrices are invertible and swap rows and columns. The matrix $a_{ij}(b)$ which is identity plus $b$ in the $(i,j)$ position is invertible, and adds a multiple of $b$ times a row/column to a row/column. Finally, we can take an identity matrix plus a row containing arbitrary things, which is still invertible. That is, $\sum_{\substack{i=1 \\ i \neq j}}^{|X|} a_{ij}(b_i)$ is invertible and will add a multiple of column $j$ to column $i$ for all $i$.

So putting A into this form $\begin{pmatrix} a_1 & & & & & & \\ & a_2 & & & & & \\ & & \ddots & & & & \\ & & & a_n & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$ by using maps $A \mapsto A' = PAQ$ comes down to figuring out whether we could put it in that form by using row operations on A. Since we showed that if A' = PAQ, $M_A = M_{A'}$, we have that M is "associated with" a matrix of this form, $\begin{pmatrix} a_1 & & & & & & \\ & a_2 & & & & & \\ & & \ddots & & & & \\ & & & a_n & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$, and so we can find the structure of M.

### Step 3

We need to show that given any matrix A, we can put it in this form $\begin{pmatrix} a_1 & & & & & & \\ & a_2 & & & & & \\ & & \ddots & & & & \\ & & & a_n & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$.

## Jordan Canonical Form