Text in purple = things that Prof. Dror Bar Natan said in class.

## Monday, October 27th

Claim: $M_{n\times n}(R[x]) \cong (M_{n\times n}(R))[x]$.

i.e. "matrices $\overline{w}$ entries as polynomials" = "polynomials $\overline{w}$ coefficients as matrices".

$$\left\{\begin{pmatrix} \sum a_{11_k}x^k & \cdots & \sum a_{1n_k}x^k \\ \vdots & & \vdots \\ \sum a_{n1_k}x^k & \cdots & \sum a_{nn_k}x^k \end{pmatrix}\right\} \qquad \left\{ \sum A_k x^k : \begin{array}{l} A_k \in M_{n\times n}(R) \\ A_k = (a_{ij_k}) \end{array} \right\}$$

$$= \{(\textstyle\sum a_{ij_k}x^k)\}$$

The map is to map coefficients to coefficients.

Caley-Hamilton Theorem

Cayley-Hamilton: "A matrix annihilates its characteristic polynomial"

Let $A \in M_{n\times n}(R)$ $R$ is a commutative ring.

$$R[t] \ni \chi_A(t) := \det(\underbrace{tI - A})$$

$$= \sum a_k t^k \qquad \underbrace{\begin{pmatrix} t-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & t-a_{22} & \cdots & -a_{22} \\ \vdots & & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & t-a_{nn} \end{pmatrix}}_{\in M_{n\times n}(R[t])}$$

$$\det(a_{ij}) = \sum_{\sigma} (-1)^{\sigma} \prod_{i=1}^{n} a_{i\sigma j}$$
$a_{ij} \in S$

Claim: $\chi_A(A) = 0$. i.e. $\sum a_k A^k = \chi_A(A) = 0$.

Wrong Proof #1:

Diagonalize matrix A, so the entries on the diagonal are the eigenvalues. Since the characteristic polynomials annihilates eigenvalues, it follows.

This is not our proof since we haven't talked about diagonalization, and the ring can be any commutative ring, so we can't diagonalize, and we can't use eigenvalues and eigenvectors.

Wrong Proof #2:

Wrong Proof: $\chi_A(A) = \det(AI - A)$
$= \det(0) = 0$.

You're putting a matrix in a matrix.
The LHS is a matrix and the RHS a scalar so the evaluation makes no sense.
We also didn't use properties of determinant, so this would also be true for the characteristic polynomial defined by trace:

Basically, it's saying that if we could just sub in A into det (tI – A), then we could also sub in A into tr (tI – A), and then the calculation doesn't make sense.

Facts needed for the correct proof:

Definition of Adj A:

Aside: Adj $A$ = "transpose of matrix of minors"
$= ((-1)^{i+j} \cdot A_{ji})_{ij}$ $\quad A_{ji} = \det(A)_{j,i}$ removing row $j$ and column $i$.

Fact about adj A:

$A \cdot \operatorname{adj} A = \operatorname{adj} A \cdot A = \det(A) \cdot I$. over any commutative $R$.

You should have seen this proof in previous courses. The proof of this fact is entirely algebraic, and it doesn't use anything except for addition and multiplication. The entries of A adj A can be reinterpreted as the determinants of the original matrix minus the row of I and column of j and replaced by other things. It's entirely algebra, so it's true over any commutative ring R.

Correct proof:

in $M_{n\times n}(R[t])$ $\qquad$ in $M_{n\times n}(R)[t]$

$$\det(tI-A)\cdot I = \operatorname{adj}(tI-A)(tI-A) = \left(\sum B_i t^i\right)(tI-A)$$

The second equality there is from the isomorphism $M_{n\times n}(R[x]) \cong (M_{n\times n}(R))[x]$.

## Monday, November 10th

Direct Sums

2 Definitions: The “set” definition (where addition and scalar multiplication is defined in the obvious way) and the category theory definition using universal property.

Defining the obvious map for a finitely generated module, R^n -> M:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mapsto \sum a_i g_i \qquad M = \operatorname{im}\pi \cong R^n/\ker\pi.$$

$$R^n \overset{\pi}{\twoheadrightarrow} M \qquad \ker\pi = \langle r_x : x \in X \rangle \rightarrow \text{not claiming finite.}$$

$$a \mapsto \sum a(x) r_x$$

Let X be a generating set for ker pi, so that any element in ker pi can be written as rx for some r \in R and x \in X.

Defining another map from X -> R:

$$\{a : X \to R;\ a(x) \neq 0 \text{ for finitely many } x\text{'s}\} = R^X \xrightarrow{A} R^n \overset{\pi}{\twoheadrightarrow} M.$$

Explaining this map in details:

$R^X$ = $\{a : X \to R;\ a(x) \neq 0 \text{ for finitely many } x\text{'s}\}$

We have a map A: $R^X$ -> R^n by defining A(b) = $\sum_{x \in X} b(x)x.$, where b is in R^x. This sum is finite because b(x) \neq 0 for finitely many x’s, and $\sum_{x \in X} b(x)x.$ is in R^n because b(x) is in R and x is in ker pi (which is in R^n), so $\sum_{x \in X} b(x)x.$ is a sum of elements in R^n.

$$\operatorname{im} A = \ker\pi \cong R^n/\operatorname{im} A.$$

Since X is a generating set for ker pi, the image of A is ker pi.

Ker pi is isomorphic to R^n/im A:

By the first isomorphism theorem, R^n/ker pi = M, so we also know that R^n/im A = M. I think $\ker\pi \cong R^n/\operatorname{im} A.$ is a typo, since this is what it’s supposed to look like:

$$R^X \xrightarrow{A} R^n \rightsquigarrow M := R^n / \operatorname{im} A.$$

A can be interpreted as an $n \times X$ matrix
finite
finite rows, infinitely many columns.

$$R^X = \langle e_x \rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ x \\ 0 \\ \vdots \end{pmatrix} x$$

A can be interpreted as an n x X matrix because A maps R^|X| to R^n. An n x X matrix maps something that's |X| dimensional to something that's n dimensional. Furthermore, in each row, there are only finitely many non-zero entries, since anything in R^X only has finitely many non-zero entries (so if we take A(e_x) for each x, we would be summing up only finitely many non-zero entries).

Furthermore, every $n \times X$ matrix $A$ defines a finitely generated module.

The finitely generated module is just the image of the matrix A (i.e., the column space), then projected by the map pi.

Examples: $A = (1) \rightsquigarrow M = R^1 / \operatorname{im} 1 = \{0\}.$

$A = (a) \rightsquigarrow M = R^1 / \operatorname{im} a = R / \langle a \rangle$

$A = (0) \rightsquigarrow M = R^1 / \operatorname{im}(0) = R / \{0\} = R.$

4 $C = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$ $M_C = M_A \oplus M_B.$

## Thursday November 13

Every f.g module is $M_A$ for some $A$.
$M$ is f.g $\Rightarrow \phi : R^n \twoheadrightarrow M$.
$\Rightarrow M \cong R^n / \ker \varphi$
Take $X = \ker \varphi$.
$R^X \to R^n$ : by $e_x \mapsto x$

Last time, we noted that A defines a finitely generated module, and this is the converse. Given a finitely generated module, take X = ker pi (where pi is the obvious projection map). Then define A: R^X -> R^n by mapping the basis elements of X to itself (since we took the generating set of ker pi X to be the whole set ker pi, it makes sense).

Claim:
$$\begin{tikzcd} R^X \arrow[r, "A"] & R^n \arrow[d, "P"] \\ R^X \arrow[u, "Q"] \arrow[r, "A'"'] & R^n \end{tikzcd}$$

$P \in M_{n \times n}(R)$   $Q \in M_{X \times X}(R)$

If P and Q are invertible, then $M_A = M_{A'}$

We would like to show that if we had such a commutative diagram, then the modules that are generated are equal.

Proof:
$$\begin{tikzcd} R^X \arrow[r, "A"] & R^n \arrow[d, "P"] \arrow[r] & R^n/\mathrm{im}A = M_A \arrow[d, dashed, shift left, "\rho"] \\ R^X \arrow[u, "Q"] \arrow[r, "A'"'] & R^n \arrow[r] & R^n/\mathrm{im}A' = M_{A'} \arrow[u, shift left, "\lambda"] \end{tikzcd}$$

$\rho$ defined w/ P and is well defined
$\lambda$ defined w/ $P^{-1}$ and is well defined.

(I'll go through the proof later)

## Jordan Canonical Form