MAT240 Sept. 26, 2006.

Examples of Vector Spaces

0. $\{0\}$ is a v.s. over any field.
1. F^n $(a_1, \ldots a_n)$ $a_i \in F$ $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$
2. $M_{m\times n}(F)$ m = rows n = columns $\begin{pmatrix} a_{11} & & a_{1n} \\ a_{m1} & & a_{mn} \end{pmatrix}$ $M_{m\times 1}(F) = F^m$
3. $\mathbb{C}/\mathbb{R}$, $\mathbb{R}/\mathbb{Q}$, ...
4. $\mathcal{F}(S, F)$ S: set

$\mathbb{C}$ is a vector space over $\mathbb{R}$.
↓
always do bigger over smaller, except in 0.

In matrices,

$$\begin{array}{c|c} & \text{col } j \\ \hline \text{row } i & \begin{pmatrix} & & \\ & a_{ij} & \\ & & \end{pmatrix} \end{array}$$

5. Polynomials eg. $7x^3 + 9x^2 - 2x + \pi$.

Let F be a field
$$P(F) = \left\{ \sum_{i=0}^{n} a_i x^i \; ; \; n \in \mathbb{Z},\ n \geq 0,\ \forall i\ a_i \in F \right\}$$

Addition of polynomials is defined in the expected way.
$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{m} b_i x^i = \sum_{i=0}^{\max(m,n)} (a_i + b_i) x^i$$
$\max(m,n) \to$ we want the bigger of m or n, or really until we run out.

Thm 1 Cancellation law for vector spaces

If in a vector space,
$$x + z = y + z$$
Then $x = y$

Pf: add w to both sides of the given equation, where w is an element for which $z + w = 0$. (exists by VS 4)

$$(x+z)+w = (y+z)+w$$
$$x+(z+w) = y+(z+w) \qquad \text{by VS2}$$
$$x+0 = y+0 \qquad \text{by choice of } w$$
$$x = y \qquad \text{by VS3.}$$

$\square$.

Thm 2 <u>0 is unique</u>: If some element $z \in V$ satisfies $x+z=x$ for some $x \in V$ then $z=0$.

Pf:

$$x+z = x+0 \qquad (0 \text{ exists})$$
$$z+x = 0+x \qquad (\text{commutativity})$$
$$z = 0 \qquad (\text{cancellation law})$$

Thm 3 <u>Negatives are unique</u>: I.e.,
If $x+y=0$ and $x+z=0$
then $y=z$.

Pf. Use cancellation law

So the notation $(-x)$ is useable because there is only one negative to x. And so, subtraction makes sense.

Thm 4
a) $0_F x = 0_v$ $\qquad 0 = 0$ from field (scalar)
b) $a \cdot 0_v = \mathbf{0}_V$ $\qquad \mathbf{0} = 0$ vector
c) $(-a)x = a(-x) \stackrel{?}{=} -(ax)$

Thm 5 If x_i $i=1,\dots,n$ are in V then "$\sum x_i$" $= x_1 + x_2 + \dots + x$ makes sense whichever way you parse it.

(Follows from VS1 and VS2) $\rightarrow$ associativity & commutativity.

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Aside: (Soccer match ends in score 3-3. Team B never led. 1. How many histories could this match have? 2. Why did I ask? 3. Find a formula $3 \to n$)

<u>Def'n</u> Let V be a vector space.
A subspace of V (vector subspace) (sub-vector space) is a subset W of V which is a vector space in itself under the operations it inherits for V.

<u>Thm.</u> $W \subset V$ is a subspace of V iff
1. $\forall x, y \in W \quad x+y \in W$
2. $\forall a \in F \ \forall x \in W \quad ax \in W$.
3. $0 \in W$ (need to put in b/c can't have empty set)

<u>Pf:</u> <u>Part 1</u> $\Rightarrow$
Assume W is a subspace
if $x, y \in W$ then $x+y \in W$ because W is a v.s. in itself, likewise for $a \cdot w$.

<u>Part 2</u> $\Leftarrow$
Assume $W \subset V$ for which $x, y \in W \Rightarrow x+y \in W$.
$x \in W$ $a \in F \Rightarrow ax \in W$.
Need to show that W is a v.s., + & · are clearly defined on W so we just need to check VS1 - VS8.

VS1 holds in V hence in W
VS2 same
VS5 - VS8 same.

VS3 $\exists\, 0 \in V \quad x+0 = x$
Pick any $x \in W \quad 0 = 0 \cdot x \in W$ by 2.

VS4. Given x in W, take $y = (-1) \cdot x \in W$ and $x+y = 0$.

Examples

$$\begin{bmatrix} 1 & 2 & 5 & 8 \\ 2 & & -7 & \\ 5 & -7 & & \\ 8 & & & \end{bmatrix}$$

Reflection

Def If $A \in M_{m\times n}(F)$ the "transpose of A," A^t is the matrix $(A^t)_{ij} := A_{ji}$

e.g. $\begin{bmatrix} 2 & 3 & \pi \\ 7 & 8 & -2 \end{bmatrix}^t = \begin{bmatrix} 2 & 7 \\ 3 & 8 \\ \pi & -2 \end{bmatrix}$

Then 1. $A^t \in M_{n\times m}(F)$
2. $(A^t)^t = A$
3. $(A+B)^t = A^t + B^t$
4. $(cA)^t = c(A^t)$

Def. $A \in M(F)$ is called "symmetric" if $A^t = A$

Claim $V = M_{n\times m}(F)$ a v.s.

let $W = \{\text{symmetric } A\text{'s in } V\} = \{A \in V : A^t = A\}$
then W is a subspace of V

Pf. 1. Need to show that if $A \in W$ & $B \in W$ then $A+B \in W$.
$A^t = A$, $B^t = B$
$(A+B)^t = A^t + B^t = A + B$
So $A+B \in W$.

W, 2. $cA \in W$: $(cA)^t = cA^t = cA \Rightarrow cA \in W$

3. $0_m = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow 0^t = 0$ So $0 \in W$.

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Example 2.

$V = M_{m\times n}(F)$ "the trace of A"

$A=(A_{ij})$ $\quad trA = \sum_{i=1}^{n} A_{ii}$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$tr = 1+5+9$
$= 15$

Properties of tr.

1. $tr\, 0_m = 0$
2. $tr(A+B) = tr(A) + tr(B)$
3. $tr(cA) = c\, trA$

Set $W = \{A\in V : tr\, A = 0\}$
$= \left\{\begin{bmatrix} 1 & 7 \\ \pi & -1 \end{bmatrix} \dots \right\}$

Claim W is a subspace

Indeed 1. $A, B \in W \Rightarrow trA = 0 = trB$
$tr(A+B) = tr(A) + tr(B) = 0+0$
So $A+B\in W$

2. $A\in W$ $\quad trA = 0.$
$tr\, cA = c\, tr\, A = c\cdot 0 = 0$
So $cA \in W$

3. $tr\, 0_m = 0$ $\quad 0_m \in W$

Example 3.

$W_3 = \{A \in M_{m\times n}(F) : \operatorname{tr} A = 1\}$

↳ Not a subspace! eg $\begin{bmatrix} 7 & 8 \\ e & -6 \end{bmatrix}$

$\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B = 1+1 = 2.$

$A, B \in W_3$ So $A+B \notin W_3$

Thm The intersection of two subspaces of the same space is always a subspace.

Assume $W_1 \subset V$ is a subspace of V,
$W_2 \subset V$ " "

then $W_1 \cap W_2 = \left\{ x : \begin{matrix} x \in W_1 \\ x \in W_2 \end{matrix} \right\}$ is a subspace.

However $W_1 \cup W_2 = \left\{ x : \begin{matrix} x \in W_1 \\ \underline{\text{or}} \\ x \in W_2 \end{matrix} \right\}$
is most often not a subspace.

Pf. 1. Assume $x, y \in W_1 \cap W_2$ that is, $x \in W_1$, $x \in W_2$
$y \in W_1$, $y \in W_2$.

$x+y \in W_1$ as $x, y \in W_1$ & W_1 is a subspace
$x+y \in W_2$ as $x, y \in W_2$ & W_2 is a subspace
So $x+y \in W_1 \cap W_2$.

2. Exercise: if $x \in W_1 \cap W_2$ then $cx \in W_1 \cap W_2$
3. $0 \in W_1$, $0 \in W_2 \Rightarrow 0 \in W_1 \cap W_2$.