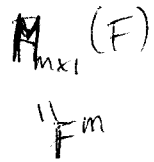
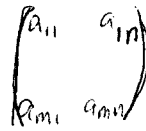


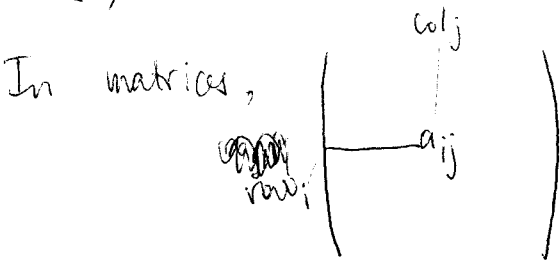
Examples of Vector Spaces

- 0. $\{0\}$ is a v.s. over any field.
- 1. F^n (a_1, \dots, a_n) $a_i \in F$
- 2. $M_{m \times n}(F)$ $m = \text{rows}$ $n = \text{columns}$



\mathbb{C} is a vector space over \mathbb{R} .
 \mathbb{C} is always do bigger over smaller, except in $\mathbb{0}$.

- 3. $\mathbb{C}/\mathbb{R}, \mathbb{R}/\mathbb{Q}, \dots$
- 4. $\mathcal{F}(S, F)$ $S = \text{set}$



- 5. Polynomials eg. $7x^3 + 9x^2 - 2x + \pi$.

Let F be a field

$$P(F) = \left\{ \sum_{i=0}^n a_i x^i : n \in \mathbb{Z}, n \geq 0, \forall i a_i \in F \right\}$$

Addition of polynomials is defined in the expected way.

$\sum_{i=0}^n a_i x^i + \sum_{i=0}^m b_i x^i = \sum_{i=0}^{\max(m,n)} (a_i + b_i) x^i$ until we run out.

Thm 1 Cancellation law for vector spaces

If in a vector space,
 $x + z = y + z$
 then $x = y$.

Pf: add w to both sides of the given equation, where w is an element for which $z + w = 0$. (exists by VS4)

$$\begin{aligned}(x+z)+w &= (y+z)+w \\ x+(z+w) &= y+(z+w) \\ x+0 &= y+0 \\ x &= y\end{aligned}$$

by VS2
by choice of w
by VS3.
□

Thm 2 0 is unique: If some element ~~the~~ ~~absorption~~ $z \in V$ satisfies $x+z=x$ for some $x \in V$ then $z=0$.

Pf:

$$\begin{aligned}x+z &= x+0 && (0 \text{ exists}) \\ z+x &= 0+x && (\text{commutativity}) \\ z &= 0 && (\text{cancellation law})\end{aligned}$$

Thm 3 Negatives are unique: I.e.,
If $x+y=0$ and $x+z=0$
then $y=z$.

Pf: Use cancellation law

So the notation $(-x)$ is useable because there is only one negative to x . And so, subtraction makes sense.

Thm 4

- $0x = 0_v$ $0 = 0$ from field (scalar)
- $a \cdot 0_v = 0_v$, $0 = 0$ vector
- $(-a)x = a(-x) = -(ax)$

Thm 5 If x_i $i=1, \dots, n$ are in V then " $\sum x_i$ " = $x_1 + x_2 + \dots + x_n$ makes sense whichever way you parse it.

(Follows from VS1 and VS2) \rightarrow associativity & commutativity.

Aside: (Soccer match ends in score 3-3
 Team B never led. 1. How many histories could this match have?
 2. Why did I ask? 3. Find a formula $3 \rightarrow n$)

Def'n Let V be a vector space.

A subspace of V is a subset W of V which is a vector space in itself under the operations it inherits from V .
 (vector subspace)
 (sub-vector space)

Thm. $W \subset V$ is a subspace of V iff

1. $\forall x, y \in W \quad x+y \in W$

2. $\forall a \in F \quad \forall x \in W \quad ax \in W$

3. $0 \in W$ (need to put in b/c can't have empty set)

Pf. Part 1 \Rightarrow

Assume W is a subspace

if $x, y \in W$ then $x+y \in W$ because W is a v.s. in itself,
likewise for $a \cdot w$.

Part 2 \Leftarrow

Assume $W \subset V$ for which $x, y \in W \Rightarrow x+y \in W$,

$x \in W \quad a \in F \Rightarrow ax \in W$.

Need to show that W is a v.s., $+$ & \cdot are clearly defined on W so we just need to check VS1-VS8.

VS1 holds in V hence in W

VS2 same

VS5-VS8 same.

VS3 $\exists 0 \in V \quad x+0=x$

Pick any $x \in W \quad 0 = 0 \cdot x \in W$ by 2.

VS4. Given x in W , take $y = (-1) \cdot x \in W$ and $x+y=0$.

Examples

$$\begin{bmatrix} 2 & 5 & 8 \\ 2 & & -7 \\ 5 & -7 & \\ 8 & & \end{bmatrix}$$

Reflection

Def If $A \in M_{m \times n}(F)$ the "transpose of A ", A^t is the matrix $(A^t)_{ij} := A_{ji}$

eg. $\begin{bmatrix} 2 & 3 & \pi \\ 7 & 8 & -2 \end{bmatrix}^t = \begin{bmatrix} 2 & 7 \\ 3 & 8 \\ \pi & -2 \end{bmatrix}$

- Then
1. $A^t \in M_{n \times m}(F)$
 2. $(A^t)^t = A$
 3. $(A+B)^t = A^t + B^t$
 4. $(cA)^t = c(A^t)$

Def. $A \in M(F)$ is called "symmetric" if $A^t = A$

Claim $V = M_{n \times m}(F)$ a v.s.

let $W = \{\text{symmetric } A \text{ in } V\} = \{A \in V : A^t = A\}$

then W is a subspace of V

Pf. 1. Need to show that if $A \in W \times B \in W$ then $A+B \in W$.

$$A^t = A, B^t = B$$

$$(A+B)^t = A^t + B^t = A+B$$

So $A+B \in W$.

W, 2. $cA \in W : (cA)^t = cA^t = cA \Rightarrow cA \in W$

3. $O_m = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow O^t = O$ So $O \in W$.

Example 2.

$V = M_{m \times n}(F)$ "the trace of A "

$A = (A_{ij})$ $\text{tr} A = \sum_{i=1}^n A_{ii}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\text{tr} = 1 + 5 + 9 = 15$$

Properties of tr .

1. $\text{tr} O_m = 0$
2. $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$
3. $\text{tr}(cA) = c \text{tr} A$

Set $W = \{ A \in V : \text{tr} A = 0 \}$
 $= \left\{ \begin{bmatrix} 1 & 7 \\ \dots & \dots \end{bmatrix} \dots \right\}$

Claim W is a subspace

- Indeed
1. $A, B \in W \Rightarrow \text{tr} A = 0 = \text{tr} B$
 $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B) = 0 + 0$
 So $A+B \in W$
 2. $A \in W \quad \text{tr} A = 0$.
 $\text{tr}(cA) = c \text{tr} A = c \cdot 0 = 0$.
 So $cA \in W$
 3. $\text{tr} O_m = 0 \quad O_m \in W$

Example 3.

$$W_3 = \{A \in M_{m \times n}(F) : \text{tr } A = 1\}$$

↳ Not a subspace! eg $\begin{bmatrix} 7 & 8 \\ e & -6 \end{bmatrix}$

$$\text{tr}(A+B) = \text{tr } A + \text{tr } B = 1+1=2.$$

$$A, B \in W_3 \quad \text{So } A+B \notin W_3$$

Thm The intersection of two subspaces of the same space is always a subspace.

Assume $W_1 \subset V$ is a subspace of V ,

$W_2 \subset V$ " " "

then $W_1 \cap W_2 = \left\{ \begin{array}{l} x : x \in W_1 \\ \quad \quad x \in W_2 \end{array} \right\}$ is a subspace.

However $W_1 \cup W_2 = \left\{ \begin{array}{l} x : x \in W_1 \\ \quad \text{or} \\ \quad \quad x \in W_2 \end{array} \right\}$

is most often not a subspace.

Pf. 1. Assume $x, y \in W_1 \cap W_2$ that is, $x \in W_1, x \in W_2$
 $y \in W_1, y \in W_2$.

$x+y \in W_1$ as $x, y \in W_1$ & W_1 is a subspace

$x+y \in W_2$ as $x, y \in W_2$ & W_2 is a subspace

So $x+y \in W_1 \cap W_2$.

2. Exercise: if $x \in W_1 \cap W_2$ then $cx \in W_1 \cap W_2$

3. $0 \in W_1, 0 \in W_2 \Rightarrow 0 \in W_1 \cap W_2$.