Problem Set 16 — MAT257

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Disclaimer—This page has been typeset by a student as a *convenient* consolidation of the homework problems. There inevitably will be mistakes; always defer to the official handout!

Problems marked with $*$ are to be submitted for credit.

1 Munkres §30 (pp.260–262)

1. Let A be open in $\mathbb{R}^n$.

   (a) Show that $\Omega^k(A)$ is a vector space.

   (b) Show that the set of all $\mathcal{C}^\infty$ vector fields on A is a vector space.

$*$ 2. Consider the forms

$$\omega = xy\,dx + 3\,dy - yz\,dz$$
$$\eta = x\,dx - yz^2\,dy + 2x\,dz,$$

in $\mathbb{R}^3$. Verify by direct computation that

$$d(d\omega) = 0,$$
$$d(\omega \wedge \eta) = (d\omega) \wedge \eta - \omega \wedge d\eta.$$

3. Let ω be a k-form defined in an open set A of $\mathbb{R}^n$. We say that ω vanishes at $\mathbf{x}$ if $\omega(\mathbf{x})$ is the zero tensor.

   (a) Show that if ω vanishes at each $\mathbf{x}$ in a neighbourhood of $\mathbf{x}_0$, then $d\omega$ vanishes at $\mathbf{x}_0$.

   (b) Give an example to show that if ω vanishes at $\mathbf{x}_0$, then $d\omega$ need not vanish at $\mathbf{x}_0$.

$*$ 4. Let $A = \mathbb{R}^2 \setminus \{\mathbf{0}\}$; consider the 1-form in A defined by the equation

$$\omega = \frac{x\,dx + y\,dy}{x^2 + y^2}.$$

   (a) Show that ω is closed.

   (b) Show that ω is exact on A.

5. Prove the following:

Theorem. Let $A = \mathbb{R}^2 \setminus \{\mathbf{0}\}$; let
$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$
in A. Then ω is closed, but not exact, in A.

Proof.

(a) Show that ω is closed.

(b) Let B consist of $\mathbb{R}^2$ with the non-negative x-axis deleted. Show that for each $(x, y) \in B$, there is a unique t with $0 < t < 2\pi$ such that
$$x = \sqrt{x^2 + y^2} \cdot \cos t$$
$$y = \sqrt{x^2 + y^2} \cdot \sin t;$$
denote this value of t by $\phi(x, y)$.

(c) Show that ϕ is of class $\mathcal{C}^\infty$. [*Hint:* The inverse sine and inverse cosine functions are $\mathcal{C}^\infty$ on the interval $(-1, 1)$.]

(d) Show that $\omega = d\phi$ in B. [*Hint:* We have $\tan\phi = y/x$ if $x \neq 0$ and $\cot\phi = x/y$ if $y \neq 0$.]

(e) Show that if g is a closed 0-form in B, then g is constant in B. [*Hint:* Use the mean-value theorem to show that if $\mathbf{a}$ is the point $(-1, 0)$ of $\mathbb{R}^2$, then $g(\mathbf{x}) = g(\mathbf{a})$ for all $\mathbf{x} \in B$.]

(f) Show that ω is not exact in A. [*Hint:* If $\omega = df$ in A, then $f - \phi$ is constant in B. Evaluate the limit of $f(1, y)$ as y approaches 0 through the positive and negative values.]

6. Let $A = \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Let m be a fixed positive integer. Consider the following $n-1$ form in A:
$$\eta = \sum_{i=1}^{n} (-1)^{i-1} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$
where $f_i(\mathbf{x}) = x_i / \|\mathbf{x}\|^m$, and where $\widehat{dx_i}$ means that the factor dx_i is to be omitted.

(a) Calculate $d\eta$.

(b) For what values of m is it true that $d\eta = 0$?

∗ 7. Prove the following, which expresses d as a generalized "directional derivative":

Theorem. Let A be open in $\mathbb{R}^n$; let ω be a $k-1$ form in A. Given $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$, define
$$h(\mathbf{x}) = d\omega(\mathbf{x})\big((\mathbf{x}; \mathbf{v}_1), \ldots, (\mathbf{x}; \mathbf{v}_k)\big),$$
$$g_j(\mathbf{x}) = d\omega(\mathbf{x})\big((\mathbf{x}; \mathbf{v}_1), \ldots, \widehat{(\mathbf{x}; \mathbf{v}_j)}, \ldots, (\mathbf{x}; \mathbf{v}_k)\big),$$
where $\widehat{a}$ means that the component a is to be omitted. Then
$$h(\mathbf{x}) = \sum_{j=1}^{k} (-1)^{j-1} Dg_j(\mathbf{x}) \cdot \mathbf{v}_j.$$

Proof.

(a) Let $X = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{bmatrix}$. For each j, let $Y_j = \begin{bmatrix} \mathbf{v}_1 & \cdots & \widehat{\mathbf{v}_j} & \cdots & \mathbf{v}_k \end{bmatrix}$. Given $(i, i_1, \ldots, i_{k-1})$, show that
$$\det X(i, i_1, \ldots, i_{k-1}) = \sum_{j=1}^{k} (-1)^{j-1} v_{ij} \det Y_j(i, i_1, \ldots, i_{k-1}).$$

(b) Verify the theorem in the case $\omega = f\,dx_I$.

(c) Complete the proof.

2 Munkres §31 (pp.265–266)

1. Prove Theorems 31.1 and 31.2.
2. Note that in the case $n = 2$, Theorem 31.1 gives us two maps α_1 and β_1 from the vector fields to 1-forms. Compare them.
3. Let A be an open set in $\mathbb{R}^3$.
   (a) Translate the equation $d(d\omega) = 0$ into two theorems about vector and scalar fields in $\mathbb{R}^3$.
   (b) Translate the condition that A is homologically trivial in dimension k into a statement about vector and scalar fields in A. Consider the cases $k = 0, 1, 2$.

* 4. For $\mathbb{R}^4$, there is a way of translating theorems about forms into more familiar language, if one allows oneself to use matrix fields as well as vector fields and scalar fields. We outline it here. The complications involved may help you understand why the language of forms was invented to deal with $\mathbb{R}^n$ in general.

A square matrix B is said to be **skew-symmetric** if $B^T = -B$. Let A be an open set in $\mathbb{R}^4$. Let $S(A)$ be the set of all $\mathcal{C}^\infty$ functions H mapping A into the set of 4×4 skew-symmetric matrices. If $h_{ij}(\mathbf{x})$ denotes the entry of $H(\mathbf{x})$ in row i and column j, define $\gamma_2 : S(A) \to \Omega^2(A)$ by the equation

$$\gamma_2(H) = \sum_{i<j} h_{ij}(\mathbf{x}) dx_i \wedge dx_j.$$

   (a) Show that γ_2 is a linear isomorphism.
   (b) Let $\alpha_0, \alpha_1, \beta_3, \beta_4$ be defined as in Theorem 31.1. Define operators "twist" and "spin" as in the following diagram:

   * [See Munkres, p.266.]

   such that

$$d \circ \alpha_1 = \gamma_2 \circ \text{twist},$$
$$d \circ \gamma_2 = \beta_3 \circ \text{spin}.$$

   (The operators are facetious analogues in $\mathbb{R}^4$ of the operator "curl" in $\mathbb{R}^3$.)

5. The operators grad, curl, and div, and the translation operators α_i and β_j, seem to depend on the choice of a basis in $\mathbb{R}^n$, since the formula defining them involve the components of the vectors involved relative to the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$. However, they in fact depend only on the inner product in $\mathbb{R}^n$ and the notion of right-handedness, as the following exercise shows.

   Recall that the k-volume function $V(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ depends only on the inner product in $\mathbb{R}^n$. (See the exercises of §21.)

   (a) Let $F(\mathbf{x}) = \big(\mathbf{x};\, f(\mathbf{x})\big)$ be a vector field defined in an open set of $\mathbb{R}^n$. Show that $\alpha_1 F$ is the unique 1-form such that
$$\alpha_1 F(\mathbf{x}) = (\mathbf{x};\, \mathbf{v}) = \langle f(\mathbf{x}), \mathbf{v} \rangle.$$
   (b) Let $G(\mathbf{x}) = \big(\mathbf{x};\, g(\mathbf{x})\big)$ be a vector field defined in an open set of $\mathbb{R}^n$. Show that $\beta_{n-1}G$ is the unique 1-form such that
$$\beta_{n-1}G(\mathbf{x})\big((\mathbf{x};\, \mathbf{v}_1), \ldots, (\mathbf{x};\, \mathbf{v}_{n-1})\big) = \epsilon \cdot V\big(g(\mathbf{x}), \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\big),$$
   where $\epsilon = +1$ if the frame $\big(g(\mathbf{x}), \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\big)$ is right-handed, and $\epsilon = -1$ otherwise.

(c) Let h be a scalar field defined in an open set of $\mathbb{R}^n$. Show that $\beta_n h$ is the unique n-form such that

$$\beta_n h(\mathbf{x})\big((\mathbf{x};\,\mathbf{v}_1),\ldots,(\mathbf{x};\,\mathbf{v}_n)\big) = \epsilon \cdot h(\mathbf{x}) \cdot V(\mathbf{v}_1,\ldots,\mathbf{v}_n),$$

where $\epsilon = +1$ if $(\mathbf{v}_1,\ldots,\mathbf{v}_n)$ is right-handed, and $\epsilon = -1$ otherwise.

(d) Conclude that the operators grad and div (and curl if $n = 3$) depend only on the inner product in $\mathbb{R}^n$ and the notion of right-handedness in $\mathbb{R}^n$. [*Hint:* The operator d depends only on the vector space structure of $\mathbb{R}^n$.]

3 "Ponder..."

Challenge! Make precise and prove Dror's assertion from class, that if $\omega \in \Omega^k(\mathbb{R}^n)$ and $\xi_1, \ldots \xi_{k+1} \in T_x\mathbb{R}^n$, then

$$d\omega(\xi_1,\ldots,\xi_{k+1}) = \lim_{\epsilon\to 0} \frac{1}{\epsilon^{k+1}}\omega\big(\partial(\epsilon P)\big),$$

where $\partial(\epsilon P)$ denotes the boundary of the parallelepiped spanned by $\epsilon\xi_1,\ldots,\epsilon\xi_{k+1}$.