

PSet 11: Partial Solutions

DISCLAIMER: I cannot claim that what I have written here constitutes a perfect solution. Certainly some mistakes are present; hopefully these mistakes aren't too severe. I hope that my answers may serve as a guide to you when studying for the final exam.

Problem 17.6. Let $U = \{(x, y, z) : x, y, z \geq 0 \text{ and } x + y + z \leq 1\}$, and let $g : U \rightarrow \mathbb{R}^3$ be the linear transformation defined by $g(x, y, z) = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x - z, 2x + y + z, 3x + 2y + z)$. The set $U \subset \mathbb{R}^3$ may be visualized as follows:

Now, $(f \circ g)(x, y, z) = x - z + 4x + 2y + 2z - 3x - 2y - z = 2x$. We note that g is \mathcal{C}^1 , as its components are \mathcal{C}^1 . We also observe that g maps this set U onto the tetrahedron S (in particular, it maps the unit vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ to the vectors $(1, 2, 3)$, $(0, 1, 2)$, $(-1, 1, 1)$, respectively). Moreover, since Dg is non-singular on U , then by the Inverse Function Theorem we have that g is invertible, and that the inverse function g^{-1} is \mathcal{C}^1 . Now, by the Change of Variables Theorem, $\int_S f$ exists if and only if $\int_U (f \circ g) |\det Dg|$ exists, and that these integrals are equal in this case.

$$\begin{aligned} \int_U (f \circ g) |\det Dg| &= \int_U 2x && \text{as computed above} \\ &= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} 2x |\det Dg| dx dy dz \end{aligned}$$

The second line follows from Fubini's Theorem, and from the fact that the sloping surface is given by $x + y + z = 1$ so that when y, z are fixed, x can range from 0 to $1 - y - z$, and when z is fixed, y can range from 0 to $1 - z$. Now,

$$|\det Dg| = \left| \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} - \det \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \right| = 2$$

Hence,

$$\begin{aligned} \int_U (f \circ g) |\det Dg| &= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} 4x dx dy dz \\ &= \int_0^1 \int_0^{1-z} 4(1 - y - z)^2 dy dz \\ &= \int_0^1 \frac{2}{3} (y + z - 1)^3 \Big|_0^{1-z} dz \\ &= \int_0^1 \frac{-2}{3} (z - 1)^3 dz \\ &= \frac{-1}{6} (z - 1)^4 \Big|_0^1 \\ &= \frac{1}{6} \end{aligned}$$

□

Problem 20.2. We say that a linear transformation $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ preserves volume if for every rectifiable set $S \subset \mathbb{R}^n$, we have that $h(S)$ is rectifiable and that $v(h(S)) = v(S)$. Now, consider the linear transformation

$g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $g(x) = \begin{pmatrix} 2 & 0 & O_{2,(n-2)} \\ 0 & 1/2 & \\ O_{(n-2),2} & & I_{n-2} \end{pmatrix} x$, where $O_{p,q}$ denotes the zero matrix of size $p \times q$.

Then,

$$\begin{aligned} \det Dg &= \det \begin{pmatrix} 2 & 0 & O_{2,(n-2)} \\ 0 & 1/2 & \\ O_{(n-2),2} & & I_{n-2} \end{pmatrix} \\ &= 2 \cdot \frac{1}{2} \cdot 1^{n-2} \\ &= 1. \end{aligned}$$

Hence, by Theorem 20.1, we have that g preserves n -dimensional volumes. Moreover, g is not an orthogonal transformation, by definition, as its columns do not form an orthonormal basis in \mathbb{R}^n . Hence, by Theorem 20.5b, g is not an isometry, as required. □

Problem 21.3. Let $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the function $h(x) = \lambda x$, and let $\mathcal{P} = \mathcal{P}(x_1, \dots, x_k)$ be a k -dimensional parallelepiped in \mathbb{R}^n . We claim that $v(h(\mathcal{P})) = |\lambda|^k v(\mathcal{P})$.

Proof. Let $X = (x_1 \ x_2 \ \dots \ x_k)$. Now, by definition of the volume of a k -dimensional parallelepiped in \mathbb{R}^n , we have that $v(h(\mathcal{P})) = |\det((\lambda X)^T(\lambda X))|^{1/2} = |\det(\lambda^2 X^T X)|^{1/2}$. We now prove by induction that if M is a $m \times m$ matrix, then $\det(cM) = c^m \det M$. For the base case, when $m = 1$, $\det(cM) = cM = c^1 \det M$, as M has only one entry. Now, suppose that the result holds for $(m-1) \times (m-1)$ matrices, and consider a matrix $M = (a_{ij})$ of size $m \times m$. Then by the determinant formula,

$$\begin{aligned} \det cM &= \sum_{i=1}^m (-1)^{i+1} (ca_{i1}) \det(cM_{i1}) && \text{where each minor } cM_{i1} \text{ is of size } (m-1) \times (m-1) \\ &= \sum_{i=1}^m (-1)^{i+1} (ca_{i1}) c^{m-1} \det M_{i1} && \text{by the inductive hypothesis} \\ &= c^m \sum_{i=1}^m (-1)^{i+1} a_{i1} \det M_{i1} \\ &= c^m \det M, \end{aligned}$$

as required. Hence, since $X^T X$ defined above is a $k \times k$ matrix, we have,

$$\begin{aligned} v(h(\mathcal{P})) &= |\det(\lambda^2 X^T X)|^{1/2} \\ &= |\lambda^{2k} \det(X^T X)|^{1/2} \\ &= |\lambda|^k v(\mathcal{P}). \end{aligned} \quad \square$$

Problem 22.2. We claim that the desired integral is given by $\int_{x \in A} \sqrt{1 + \sum_{i=1}^k (\partial_i f(x))^2}$.

Proof. Since α is \mathcal{C}^1 , then by the first definition in section 22 of Munkres we have that the volume of Y_α is defined as $v(Y_\alpha) = \int_A V(D\alpha)$. Now, $\alpha(x) = (x, f(x)) = (i(x), f(x))$ for all $x \in A$, where i is the identity

function. Hence, $D\alpha = \begin{pmatrix} Di \\ Df \end{pmatrix} = \begin{pmatrix} I_k \\ Df \end{pmatrix}$. So,

$$\begin{aligned}
 V(D\alpha) &= |\det(D\alpha)^T(D\alpha)|^{1/2} \\
 &= \left| \det \left(\begin{pmatrix} I_k^T & Df^T \\ & Df \end{pmatrix} \cdot \begin{pmatrix} I_k \\ Df \end{pmatrix} \right) \right|^{1/2} \\
 &= |\det(I_k \quad Df^T Df)|^{1/2} \\
 &= |1 + Df Df^T|^{1/2} && \text{by Dror's hint, that } \det(I_n + vw^T) = 1 + v^T w \quad \forall v, w \in \mathbb{R}^n \\
 &= \sqrt{1 + \sum_{i=1}^k (\partial_i f)^2}
 \end{aligned}$$

Hence,

$$\begin{aligned}
 v(Y_\alpha) &= \int_A V(D\alpha) \\
 &= \int_{x \in A} \sqrt{1 + \sum_{i=1}^k (\partial_i f(x))^2}
 \end{aligned}$$

□

Exercise 2.1. We are given the parametrization $\sigma : U \rightarrow \mathbb{R}^3$ of an orientable surface, where $U = (0, 1) \times (0, \pi)$ and σ is defined by $\sigma(u, v) = (u \cos v, u \sin v, v)$, for all $(u, v) \in U$. Since U is open in \mathbb{R}^2 , and σ is \mathcal{C}^1 , then σ is a parametrized 2-manifold in \mathbb{R}^3 . Now,

$$D\sigma(u, v) = \begin{pmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \\ 0 & 1 \end{pmatrix}.$$

Hence, by a formula derived in lecture, the area $\mathcal{A}(\sigma(U))$ is given by:

$$\begin{aligned}
 A(\sigma(U)) &= \int_U |\det(D\sigma)^T(D\sigma)|^{1/2} \\
 &= \int_U \left| \begin{pmatrix} \cos v & -\sin v & 0 \\ -u \sin v & u \cos v & 1 \end{pmatrix} \begin{pmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \\ 0 & 1 \end{pmatrix} \right|^{1/2} \\
 &= \int_U \left| \begin{pmatrix} 1 & 0 \\ 0 & u^2 + 1 \end{pmatrix} \right|^{1/2} \\
 &= \int_{(0,1) \times (0,\pi)} \sqrt{u^2 + 1} \\
 &= \int_0^\pi \int_0^1 \sqrt{u^2 + 1} \\
 &= \pi \int_0^{\pi/4} \sec^3 \theta d\theta \\
 &= \pi \left(\tan \theta \sec \theta \Big|_0^{\pi/4} - \int_0^{\pi/4} (\sec^2 \theta - 1) \sec \theta d\theta \right)
 \end{aligned}$$

by Fubini.

by the substitution $\tan \theta = u$

int. by parts with

$u = \sec \theta, dv = \sec^2 \theta$

$$\begin{aligned}
 \implies 2 \int_0^{\pi/4} \sec^3 \theta d\theta &= \tan \theta \sec \theta \Big|_0^{\pi/4} + \int_0^{\pi/4} \sec \theta d\theta \\
 \int_0^{\pi/4} \sec^3 \theta d\theta &= \frac{\sqrt{2}}{2} - 0 + \frac{1}{2} \ln(1 + \sqrt{2}) - \frac{1}{2} \ln(1) \\
 &= \frac{1}{\sqrt{2}} + \frac{1}{2} \ln(1 + \sqrt{2}).
 \end{aligned}$$

Hence, the area is given by:

$$A(\sigma(U)) = \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} \ln(1 + \sqrt{2}).$$

□