Suppose M is finitely generated w/ generators $g_1, \dots, g_n \in M$. Ie. M is an R-module,
$M = \left\{ \sum_{i=1}^{n} a_i g_i : a_i \in R \right\}$.

Consider the map $\pi : R^n \twoheadrightarrow M$, $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mapsto \sum_{i=1}^{n} a_i g_i$. π is clearly onto. Is π an R-module homo?
$\pi\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \pi\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1}^{n} a_i g_i + \sum_{i=1}^{n} b_i g_i = \sum_{i=1}^{n} (a_i + b_i) g_i = \pi\begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}$
yes. the other is also obvious.

So, $R^n / \ker \pi \cong \mathrm{im}(\pi) = M$. Now, $\ker(\pi) = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in R^n : \sum_{i=1}^{n} a_i g_i = 0 \right\} \to$ doesn't tell us much.

Let $X = \ker(\pi) \subset R^n$. For each $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in X$, write $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = r_{(b_1, \dots, b_n)^T}$, ie. write $r_x := x$. So,
$\ker(\pi) = X = \{ r_x \in R^n : x \in X \}$. $\therefore$ $\ker(\pi) = X = \langle r_x : x \in X \rangle$. (trivially)

Let $R^X = \left\{ a : X \to R \mid a(x) \neq 0 \text{ for finitely many } x\text{'s} \right\}$. (Note: this is not the usual R^X, which is the set of fns from X to R.)

↳ Here, we're thinking of X as an index set.

can think of elements of R^X as "X-tuples": $a \in R^X \Leftrightarrow a = (\underset{\in R}{a(x_1)}, \underset{\in R}{a(x_2)}, \underset{\in R}{a(x_3)}, \dots) = (a_{x_1}, a_{x_2}, a_{x_3}, \dots)$

Define a map $A : R^X \to R^n$ by
$a \mapsto \sum_{x \in X} a(x) r_x$ (ok b/c only finitely many are nonzero.)

↳ can think of $\sum_{x \in X} a(x) r_x$ in terms of matrix mult.:
If $r_x = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, $a(x) r_x = a(x) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a(x) b_1 \\ \vdots \\ a(x) b_n \end{pmatrix}$. So, $\sum_{x \in X} a(x) r_x$ "=" $(a(x_1), a(x_2), \dots) \begin{pmatrix} r_{x_1} \\ r_{x_2} \\ \vdots \end{pmatrix}$

So by construction, $\mathrm{im}(A) = X = \ker(\pi)$. Thus, $M \cong R^n / \ker(\pi) = R^n / \mathrm{im}(A) =: M_A$

$$R^X \xrightarrow{A} R^n \xrightarrow{\pi} M$$
$$a \mapsto \sum_{x \in X} a(x) r_x \qquad \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \mapsto \sum_{i=1}^{n} b_i g_i.$$

Let $J = |X|$. Since J is indexing $|X|$, $X = \{ x_\eta : \eta \in J \}$.

$\forall x_\eta \in X$, let $e_{x_\eta} \in R^X$ be the function $e_{x_\eta}(y) = \begin{cases} 0 & \text{if } y \neq x_\eta \\ 1 & \text{if } y = x_\eta \end{cases}$.

If $J = \{1, 2, \dots\}$, think of e_{x_j} as an infinite basis vector: $\begin{array}{cl} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow j^{th} \text{ spot} \\ \\ \\ \end{array} \end{array}$

Notice that $R^X = \langle e_x : x \in X \rangle$, since for any $a \in R^X$, $a(x) \neq 0$ for only finitely many x's, say $x_{\eta_1}, x_{\eta_2}, \dots, x_{\eta_k}$. So $a = \sum_{\ell=1}^{k} a(x_{\eta_\ell}) \cdot e_{x_{\eta_\ell}}$. (let $\mathrm{supp}(a) = \{ x_\eta \in X : a(x_\eta) \neq 0 \}$.)