This is the solution I gave (in assignment 3) for problem 10 in section 2.3. This is a rewritten version, and all suggestions and comments are welcome.

Problem. *Suppose a map is made by drawing n intersecting circles. Show that the regions in this map can be properly 2-colored.*

We claim that by assigning colors based on the number of circles that contain a region, the regions in the given map can be 2-colored.

Definition 1. *Given a map made by drawing n intersecting circles, we define the "vertices" to be all points of intersection of the n circles, and the "edges" to be all arcs (of the n circles) whose endpoints are two "vertices".*

Definition 2. *The boundary of a region is a set of edges bounding the region. Two different regions are said to be adjacent if there is a common edge in their boundaries.*

Lemma. *For each region* r *in the map, let* $\mathrm{f(r)}$ *be the number of circles that contain* r. *If* $\mathrm{x, y}$ *are adjacent, then*

$$|\mathrm{f(x)} - \mathrm{f(y)}| = 1.$$

Proof. For each pair of adjacent regions $\mathrm{x, y}$, *there is a common edge* $e \in \mathcal{C}$ *in their boundaries, where* $\mathcal{C}$ *is a circle. Notice that the map is planar, so any two regions do not overlap, meaning that* $\mathcal{C}$ *contains exactly one of* $\mathrm{x, y}$. *Without loss of generality, we may assume that* $\mathcal{C}$ *contains* x. *We claim that* $\mathcal{C}$ *is the only circle containing* x *that does not contain* y.

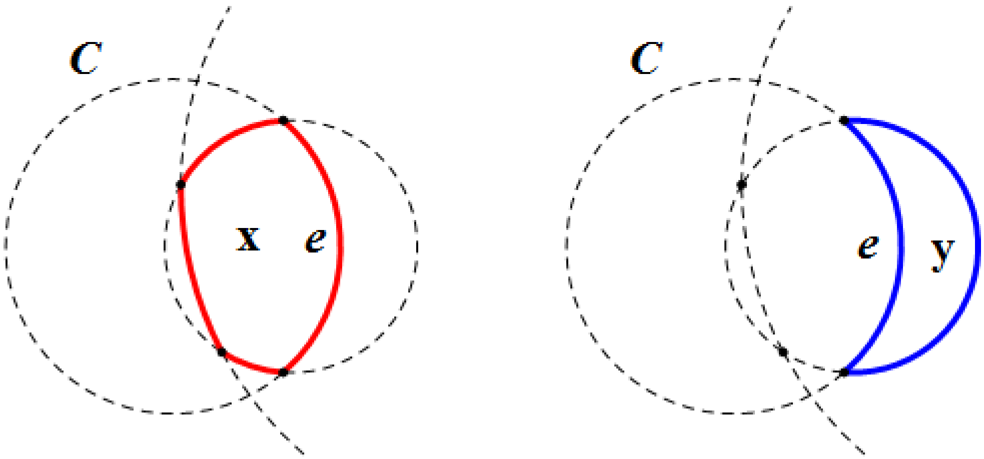

Figure 1. x, y *are adjacent.*

Indeed, assume that $\mathcal{D}$ *is a circle that contains* x *but does not contain* y, *and that* $\mathcal{D} \neq \mathcal{C}$. *The planarity of the map implies that a point on any of the n circles cannot be an interior point of any region. Therefore* $\mathrm{Int}\,(d) \notin y$ *for all arc* d *of* $\mathcal{D}$, *where* $\mathrm{Int}\,(d)$ *denotes the interior of* d, *(otherwise if* $\mathrm{Int}\,(g) \in y$ *for some arc* g *of* $\mathcal{D}$, *then any interior point of* g *will be an interior point of* y*), meaning that* $\mathcal{D}$ *must contain* y, *contradicting the definition of* $\mathcal{D}$.

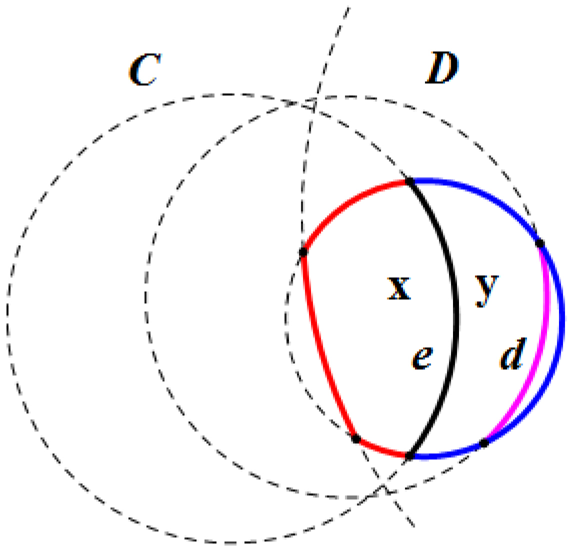

Figure 1.2. $\mathrm{Int}\,(d) \in y$ *leads to a contradiction.*

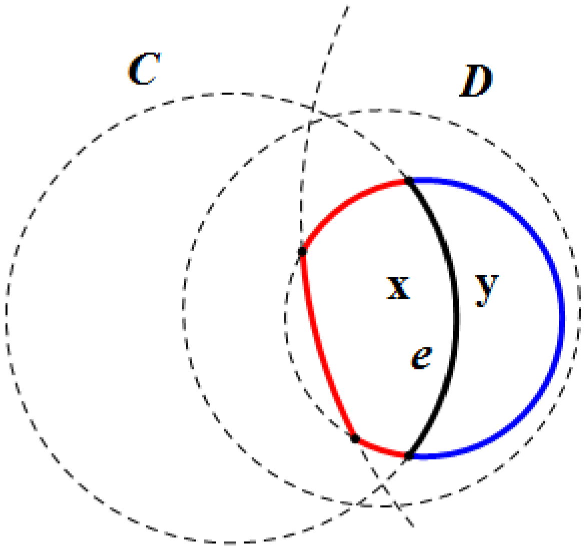

Figure 1.3. $\mathcal{D}$ *contains* y.

Consequently, every circle other than $\mathcal{C}$ *that contains* x *must contain* y. *Likewise, if* $\mathcal{C}$ *contains* y *but does not contain* x, *we know by the same reasoning that every circle other than* $\mathcal{C}$ *that contains* y *must contain* x. *Therefore the number of circles that contain* x *and the number of circles that contain* y *differ by* 1, *i.e.*

$$|\mathrm{f}(\mathrm{x}) - f(y)| = 1$$

completing the proof of lemma. □

Assume that we are given two colors that are labeled 1, 2 *respectively. Consider a region* r *in the map. If* f(r) *is odd, we color* r *with* 1. *Otherwise color* r *with* 2. *Using the lemma, we know that the coloring above is legitimate, and we are done.* □