Math 327 Homework 8

Exercise 1. *(Munkres, #2, p. 223) If instead of dividing the interval $[-r, r]$ into three equal pieces, we instead divided into pieces $[r, -qr], [-qr, qr], [qr, r], 0 < a < 1$, for what values of q does the proof of the Tietze theorem go through?*

I claim that any $q < 1/2$ works. It's a bit tedious to check in full, but what we need is

$$|g(a) - f(a)| \leq r/k, \forall a \in A$$

where $k > 1$, so that the resulting geometric series will converge (see text). The bound on $|g(a) - f(a)|$ is obtained by considering the case where a lies in each of the three preimages (under f) of our three subintervals. The "worst case" occurs when $a \notin B \cup C$ (see text); then the only bound we can get is $|g(a) - f(a)| < 2q$, so we need $2q < 1$.

Exercise 2. *(Munkres, #5a, p. 223) A space is said to have the* universal extension property *if for each triple consisting of a normal space X, a closed subset A of X, and a continuous function $f : A \longrightarrow Y$, there exists an extension of f to a continuous map of X into Y. Show $\mathbb{R}^J$ has the universal extension property.*

Given any such tuple $(X, A, f : A \longrightarrow \mathbb{R}^J)$, we may, by Tietze's Theorem, extend each coordinate of f, say $f_\alpha : A \longrightarrow \mathbb{R}, \alpha \in J$, to a continuous map $f'_\alpha : X \longrightarrow \mathbb{R}$. Reassembling these maps in the obvious way, we get a map $f' : X \longrightarrow \mathbb{R}^J$ which agrees with f on A and is continuous by definition of the product topology.

Exercise 3. *(Munkres, #1, p. 270) Let X be metric and suppose there exists $\epsilon > 0$ such that every ϵ-ball in X has compact closure. Then X is complete, but it is not true that we can reverse the order of the $\forall$ and $\exists$ quantifiers in the above statement.*

Let $\{x_n\}$ be a Cauchy sequence in X. This sequence eventually lies inside some ϵ-ball (this is obvious from the definition of Cauchy sequence), hence inside its closure, which is compact. Hence $\{x_n\}$ eventually lies inside a compact, hence complete, space, and so converges, say to $x \in X$.

Let $X = (0, 1)$, which is certainly not complete, but it's clear that given any $x \in X$ we can find ϵ_x such that $\overline{B_{\epsilon_x}(x)}$ is compact.

Exercise 4. *(Munkres, #4, p. 270) Show that the metric space (X, d) is complete iff for every nested sequence $A_1 \supseteq A_2 \supseteq \ldots$ of nonempty closed sets of X with $diam A_n \longrightarrow 0$, the intersection of all the A_n is nonempty.*

Suppose first that X is complete. Construct $\{x_n\}$ by taking $x_n \in A_n$; it's clear that this is a Cauchy sequence. Writing $x_n \longrightarrow x$, we see that $x \in A_n$ for every A_n (or $\{x_n\}$ could not converge to x), so $x \in \cap A_n$.

Conversely, let $\{x_j\}$ be a Cauchy sequence. Define $A_n = \overline{\{x_j\}_{j=n}^{\infty}}$; these sets clearly satisfy the above hypotheses, so we may produce $x \in \cap A_n$. Then $\{x_j\}$ has a subsequence converging to x, so converges to x itself (by a lemma on Cauchy sequences).

Exercise 5. *(Munkres, #5, p. 270) A map $f : X \longrightarrow X$ is a* contraction map *if there exists $\alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. If f is a contraction of a complete metric space, then f has a unique fixed point.*

Uniqueness is clear: if $f(x) = x, f(y) = y$ for distinct x, y, then $d(x, y) = d(f(x), f(y)) \leq \alpha d(x, y) < d(x, y)$, a contradiction. To show existence, we'll apply f repeatedly to some point to generate a sequence which gets arbitrarily close to a fixed point. Fix any $x \in X$. To show the sequence $\{x_n\}$ given by $x_n = f^n(x)$ is Cauchy, first note

$$d(f^m(x), f^{m+1}(x)) \leq \alpha d(f^{m-1}(x), f^m(x)) \leq \ldots \leq \alpha^m d(x, f(x)).$$

Then, assuming $n > M$,

$$\begin{aligned} d(f^n(x), f^M(x)) &\le d(f^n(x), f^{n-1}(x)) + \ldots + d(f^{M+1}(x), f^M(x)) \\ &\le \alpha^{n-1} d(x, f(x)) + \ldots + \alpha^M d(x, f(x)) \\ &\le (\alpha^M + \alpha^{M+1} + \ldots) d(x, f(x)) \\ &= \frac{\alpha^M}{1-\alpha} d(x, f(x)) \longrightarrow 0 \text{ as } M \longrightarrow \infty. \end{aligned}$$

This can easily be shown to imply that $\{x_n\}$ is Cauchy and so converges, say to $x \in X$. To show x is the desired fixed point, note that f is α-Lipschitz, hence (uniformly) continuous, so $f(x) = \lim_{n\to\infty} f(x_n) = \lim x_{n+1} = x$. Alternately,

$$\begin{aligned} d(x, f(x)) &\le d(x, x_m) + d(x_m, f(x_m)) + d(f(x_m), f(x)) \\ &\le 2d(x, x_m) + d(x_m, f(x_m)). \end{aligned}$$

Since the right hand side can be made as small as we like, it must be that $d(f(x), x) = 0$.

Exercise 6. *(Munkres, #1, p. 280) A countable product of totally bounded metric spaces is totally bounded; a countable product of compact metric spaces is compact, and this result does not depend on Tychonoff's Theorem (hence doesn't depend on the Axiom of Choice).*

Let $X = \prod_{n\in\mathbb{N}} X_n$ where each X_n is totally bounded. Recall that $D(x, y) = \sup\{\bar{d}_i(x_i, y_i)/i\}$ is a metric for the product space. Fix $\epsilon > 0$. Choose N large enough that $i \ge N$ implies $1/i < \epsilon/2$. Let $\{x_{nm}\}_{n\in\mathbb{N}, m\le k_n}$ be, for each n, a finite set of points such that for all $x_n \in X_n$ there exists x_{nm} such that $\bar{d}_n(x_n, x_{nm}) < \epsilon$. Then it's clear that for any $x \in X$ there exists some $y_j = (x_{1m_1}, x_{2m_2}, \ldots, x_{nm_n}, 0, 0, 0, \ldots) \in \prod_{n\in N, m\le k_n}\{x_{nm}\} \times 0 \times 0 \times \ldots$ such that $D(x, y_j) < \epsilon$, and that only finitely many such y_j exist. In other words, X is totally bounded.

Completeness is already done (essentially we repeat the proof of Theorem 43.4). Hence a countable product of compact metric spaces is compact.

Exercise 7. *"Problem IX." Show, without Urysohn's Lemma, that every metric space (X, d) can be embedded in a cube I^X. (Hint: given $x \in X$, what real-valued function on X comes to mind?)*

If X is empty, there is nothing to show. Otherwise, the obvious function is $f_x : X \longrightarrow [0, 1]$ given by $f_x(y) = \tilde{d}(x, y)$.

This map is clearly continuous; it is injective since $f(x)$ vanishes exactly in the xth coordinate. All that remains to show is continuity of the inverse function $f^{-1} : [0, 1]^X \longrightarrow X$. Fix any basic open set $B_\epsilon(x_0) \subset X$. If $\epsilon > 1$, this ball is all of X, so any neighbourhood of $f^{-1}(x_o)$ has image (under f^{-1}) contained in that ball. If $\epsilon \le 1$, consider the neighbourhood $U = \pi_{x_o}^{-1}([0, \epsilon)) \cap f(X)$, which is open in $[0, 1]^X$ (as $\pi_{x_0} : [0, 1]^X \longrightarrow [0, 1]$ is continuous) and contains $f(x_0)$. Now, if $x \in U$, then $f_{x_0}(x) < \epsilon$, so $x \in B_\epsilon(x_0)$. Thus for arbitrary basic $B_\epsilon(x_0)$ we've found $U \ni x_0$ such that $f^{-1}(U) \subset B_\epsilon(x_0)$, and we're done.