Analysis II (MAT257) HW7
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§11

1. Consider $\mathbb{Q}$: as a countable set, it is the union of countably many trivially measure zero sets (singleton sets) and thus is measure zero. But $\overline{\mathbb{Q}} = \mathbb{R}$ and $\mu(\mathbb{R}) = \infty$. Thus, the closure of a set and the original set can have dramatically different measures.

4. Well, $A \times B = \{(a, b) | a \in A, b \in B\}$. It follows, since the empty set is, well, empty, that $\mathbb{R}^{n-1} \times 0 = 0$. And the empty set is trivially measure zero – otherwise measure would explode since any set can be interpreted as itself, union the empty set!

6. By the definition of measure zero, for every epsilon greater than zero there exists a countable collection of rectangles $\mathcal{Q}$ such that $\sum_{Q\in\mathcal{Q}} \mu(Q) < \epsilon$. The definition provided on page 91 of the textbook does not state whether these rectangles are open or closed (it seems unlikely that they would be anything else). If they are open, we use A's compactness to find a finite subcover and consider those, giving us a finite collection of rectangles. If A is not open, we turn them into open rectangles (throw away the boundaries) and add $\delta/2$ outwards to each edge of the triangle: in the worst case, the rectangles are a single cube and our new measure is $(\sqrt[n]{\epsilon} + \delta)^n$. We take a finite subcover and then close all the rectangles. We now have a finite cover with measure $(\sqrt[n]{\epsilon} + \delta)^n$ and choose a ϵ' and δ such that the measure is less than the original ϵ.

1. Consider a partition p formed by Q, strips from the boundary of Q out a tiny ways sot that the total measure of them is ϵ and some rectangles that fill the rest of the space. Then $U(f, p) = 1*v(Q)+1*\epsilon+0$ and $L(f, p) = 1 * v(Q) + 0 * \epsilon + 0$. Then $U - L = \epsilon$. And by shrinking the the strips so that they have the measure of a smaller epsilon, we can make it arbitrarily small. So $\overline{\int}_{\mathbb{R}^n} f = \underline{\int}_{\mathbb{R}^n} f = v(Q)$.

2. (a) Let M be the maximum value of our bounded function. Then we choose a partition containing the right Q_i to cover S with a total measure of ϵ/M, thereby forcing the value of $U(f, p) = \epsilon$ ($L(f, p) = 0$, of course). So it is integrable and the value of the integral is zero.

   (b) Since f and g are bounded and S has content zero, we can create a partition that has in which S is wholly contained in a finitely many rectangles and the integral on them is less than some epsilon. Thus, the limit of our refinements of partitions will lead to S being totally irrelevant and, since f and g are the same everywhere except on S and we're using the same partitions on both, they will have the same integral. Thus, if one integral exists, the other will and they will share the same value.

   (c) Suppose we have n sets s_n each covered by a collection of rectangles with a total measure less than ϵ for an arbitrary ϵ, $\mathcal{Q}_n^\epsilon$. Then $S = \bigcup s_n$ is covered by a collection of rectangles with total measure less than ϵ, $\mathcal{Q} = \bigcup \mathcal{Q}_n^{\epsilon/n}$.

   (d) We create strips along the edges of the rectangle, thereby covering the edges with arbitrarily small measure rectangles. Since it is finite dimensional, there are only finitely many of these strips. So we can find a finite cover of the *bd* such that they have a total measure that is arbitrarily small.

   (e) Suppose that the closure of S was of some measure $m > 0$. By the definition of content zero, I could find a finite cover of closed sets with measure less than $\epsilon = m/2$. Their union is a closed set with measure less than $m/2$. This contradicts our supposed closure of S being the smallest closed set, which it must be, by definition.

   (f) Suppose that $S = \mathbb{Q} \cap [0, 1]$ had content zero. Then we could find a finite cover of closed sets with total measure less than $1/2$. But their union would be a closed set containing S with measure less than half, which sets an upper limit of $1/2$ on the measure of the closure of S. But the closure of S is $[0, 1]$ (which has measure 1) since $\mathbb{Q}$ is dense on the reals.

3. Since f is integrable, there is a collection of partitions $\mathcal{P}$ such that $\int_Q f = inf_{P\in\mathcal{P}}U(f,P) = sup_{P\in\mathcal{P}}L(f,P)$. It follows that $\mathcal{P}$ such that $\int_Q cf = inf_{P\in\mathcal{P}}U(cf,P) = c * inf_{P\in\mathcal{P}}U(f,P) = sup_{P\in\mathcal{P}}L(cf,P) = c * sup_{P\in\mathcal{P}}L(f,P) = c\int f$.

4. Since f_1 and f_2 are integrable, $\int_Q f_1 = inf_{P\in\mathcal{P}_1}U(f_1,P) = sup_{P\in\mathcal{P}_1}L(f_1,P)$ and $\int_Q f_2 = inf_{P\in\mathcal{P}_2}U(f_2,P) = sup_{P\in\mathcal{P}_2}L(f_2,P)$. If we make the partition $\mathcal{P} = \{p_1 \cap p_2 | p_1 \in \mathcal{P}_1, p_2 \in \mathcal{P}_2\}$. It follows that $infU(f_1+f_2,\mathcal{P}) = inf_{P\in\mathcal{P}_1}U(f_1,P) + inf_{P\in\mathcal{P}_2}U(f_2,P)$ and $supL(f_1+f_2,\mathcal{P}) = sup_{P\in\mathcal{P}_1}L(f_1,P) + sup_{P\in\mathcal{P}_2}L(f_2,P)$. So $\int(f_1+f_2) = \int f_1 + \int f_2$.

5. Since f is zero on the irrational numbers, we only need to consider how are partition covers the rationals. Since they are measure zero, (since we can find a measure-ϵ closed cover, eg. $\{\mathbb{Q}_n - \frac{\epsilon}{2^n}, \mathbb{Q}_n + \frac{\epsilon}{2^n}]|n \in \mathbb{N}_{>0}\}$, $\mathbb{Q}_n$ denotes the nth rational in $[0,1]$ in some ordering of them) and f is bounded (when we represent a rational number in the reduced form p/q, the minimum value of q is one and the maximum value of $1/q$ is one, which acts as a bound. So, $inf_{P\in\mathcal{P}}U(f,P) = 0$ (and trivially $sup_{P\in\mathcal{P}}L(f,P) = 0$) so $\int f = 0$.

6. Since f is integrable, there is a collection of partitions $\mathcal{P}$ such that $\int_Q f = inf_{P\in\mathcal{P}}U(f,P) = sup_{P\in\mathcal{P}}L(f,P)$. But we can cover the graph with the rectangles $\bigcup_{p\in P} p\times[infp, supp]$. But $\inf_{P\in\mathcal{P}}\bigcup_{p\in P} p\times[infp, supp] = inf_{P\in\mathcal{P}}U(f,P) - sup_{P\in\mathcal{P}}L(f,P) = 0$. So the graph is measure zero.