MAT1100HF, TERM TEST, HINTS-SOLUTIONS

Grigorios Fournodavlos
St. No.: 998805761

Problem 1

1. Standard theory, the proof is in the notes.

2. Suppose there is an element $x \in G$, whose order is a power of p, such that $x \in N_G(P)$ and $x \notin P$. It follows that $< x >\leq N_G(P)$ and since $P \lhd G$, we have $< x > P \leq N_G(P)$. The second isomorphism theorem implies that $| < x > P| = \frac{|<x>||P|}{|<x>\cap P|} = |P| \cdot \frac{|<x>|}{|<x>\cap P|} > |P|$, because $< x > \cap P$ is a proper subset of $< x >$ ($x \notin < x > \cap P$). Also, note that $| < x > P|$ is a power of p (look at the right hand of the previous relation). This means that P is not a Sylow-p subgroup of G, which is a contradiction. ■

Problem 2

1. Let $G \rtimes H$ be a semidirect product of two torsion free groups G, H. Suppose that a non trivial element $(g, h) \in G \rtimes H$ has finite order n.

   (a) If $h \neq e_H$, then $(g, h)^n = (*, h^n) = (e_G, e_H) \Rightarrow h^n = e_H$, which is a contradiction.

   (b) If $h = e_H$, then $g \neq e_G$ and $(g, h)^n = (g, e_H)^n = (g^n, e_H) = (e_G, e_H) \Rightarrow g^n = e_G$, again a contradiction.

2. By induction on the first part, it follows that the pure braid group $PB_n \cong F(n-1) \rtimes (F(n-2) \rtimes (\cdots \rtimes (F(2) \rtimes F(1))))$ is torsion free and so the only element $\beta \in PB_n$ with finite order satisfying $\beta^7 = e$ is the identity, $\beta = e$. ■

Problem 3

$(\Rightarrow)$ Let $f : G/H_1 \to G/H_2$ be a G-isomorphism. Also, let $f(H_1) = xH_2$ for some $x \in G$. Then for all $h_1 \in H_1$ it holds $xH_2 = f(H_1) = f(h_1 \cdot H_1) = h_1 \cdot f(H_1) =$

$h_1 \cdot xH_2 = h_1xH_2 \;\Rightarrow\; h_1 \cdot xH_2x^{-1} = xH_2x^{-1} \;\Rightarrow\; h_1 \in xH_2x^{-1}, \forall h_1 \in H_1 \;\Rightarrow\; H_1 \leq xH_2x^{-1}$. Further, f is bijective so $|G : H_1| = |G : H_2| \;\Rightarrow\; |H_1| = |H_2| \;\Rightarrow\; H_1 = xH_2x^{-1}$, which means that H_1, H_2 are conjugate.
($\Leftarrow$) Let H_1, H_2 be conjugate subgroups of G and $x \in G$ such that $H_1 = xH_2x^{-1}$. Define $f : G/H_1 \to G/H_2$, $f(gH_1) = gxH_2$. For $g, g' \in G$ it holds $f(gH_1) = f(g'H_1)$ $\Leftrightarrow\; gxH_2 = g'xH_2 \;\Leftrightarrow\; x^{-1}g^{-1}g'x \in H_2 \;\Leftrightarrow\; g^{-1}g' \in xH_2x^{-1} = H_1 \;\Leftrightarrow\; gH_1 = g'H_1$ and so f is well defined and injective. Also, it is obvious that f is surjective and respects the actions, which makes it a G-isomorphism. ■

Problem 4

1. $<(12), (12\cdots n)> \leq G \;\Rightarrow\; (12\cdots n)^{-1}(12)(12\cdots n) = (1n) \in G$ and $(12\cdots n)^{-k}(12)(12\cdots n)^k = (n-k+1\;\; n-k+2) \in G$, $k = 2, \cdots, n$. Thus, $(13) = (23)(12)(23) \in G$, $(14) = (34)(13)(34) \in G$ and continuing this way we obtain $(1i) \in G$, $\forall i$. It follows that $(ij) = (1i)(1j)(1i) \in G$, $\forall i, j$ and hence $G = S_n$ since every permutation can be written as a product of transpositions.

2. $<(123), (12\cdots n)> = G$, n is odd. Of course, $G \leq A_n$. In order to prove the reverse inclusion it suffices to show that $(abc) \in G\; \forall a, b, c$, which in turn is reduced in proving $(1ab) \in G$, $\forall a, b$, since $(abc) = (1cb)(1ab)(1ac)$. Again, the last statement can be reduced further to the fact that $(12a) \in G$, $\forall a$, because $(1ab) = (12b)(12a)(12a)$.

   - Similarly to the first part we have $(12\cdots n)^{-k}(123)(12\cdots n)^k \in G$, $\forall k \;\Rightarrow\; (12n), (1\;\; n-1\;\; n) \in G$ and $(n-k+1\;\; n-k+2\;\; n-k+3) \in G$, $k = 3, \cdots, n$.

   Hence, it holds $(34)(23)(123)(23)(34) = (34)(132)(34) = (142) \in G \;\Rightarrow\; (124) \in G$ $\Rightarrow\; (35)(34)(124)(34)(35) = (35)(123)(35) = (125) \in G \;\Rightarrow\; (46)(45)(125)(45)(46) = (126) \in G$ and so on we obtain $(12a) \in G$, $\forall a$. This completes the proof.

3. If n is even then of course it is not true that $G \leq A_n$, since the cycle $(12\cdots n)$ is an odd permutation. However, the same method we used in the second part can be applied in this case also, which yields that $G \supset A_n \;\Rightarrow\; G \supset A_n \cup (12\cdots n)A_n = S_n$ and therefore $G = S_n$. ■