Section 6.

9. $g: \mathbb{R} \to \mathbb{R}$ a function of C^2. Thm: $\lim_{h\to 0} \frac{g(a+h) - 2g(a) + g(a+h)}{h^2} = g''(a)$

proof: ~~Let Q... Define $\varphi(a) = (g(a) - g(a+h))/h$~~.

Define $\varphi(a) = g(a) - g(a+h)$ then $\lim_{h\to 0} \frac{\varphi(a)}{h} = g'(a)$ by def. and (because g is C^2).

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because Dg exist ~~φ is~~ in $\mathbb{R}$, φ is diff in an open interval containing the closed interval bdd by a and $a+h$.

hence $\varphi(a+h) - \varphi(a) = \varphi'(s_0) h$ for some s_0 between a and $(a+h)$.

Say $s_0 = a + \lambda h$ for some $\lambda \in [0,1]$ fixed.

then $\lim_{h\to 0} \frac{g(a+h) - 2g(a) + g(a+h)}{h^2} = \lim_{h\to 0} \frac{\varphi(a+h) - \varphi(a)}{h^2} = \lim_{h\to 0} \frac{\varphi'(s_0)\cdot h}{h^2} = \lim_{h\to 0} \frac{\varphi'(a+\lambda h)}{h} = \lim_{h\to 0} \frac{\varphi'(a)}{h}$.

But g is a function of $C^2 \Rightarrow \lim_{h\to 0} \frac{\varphi'(a)}{h} = \lim_{h\to 0} \frac{g'(a) - g'(a+h)}{h} = g''(a)$. $\square$.

(Because g is C^2, then g is twice continuously diff., so $\varphi'(a) = (g(a) - g(a+h))' = g'(a) - g'(a+h)$ and then $g'(a)$ is of C', by def. of differentation, $\lim_{h\to 0} \frac{g'(a) - g'(a+h)}{h} = g''(a)$.) $\square$.

10. $f: \mathbb{R}^2 \to \mathbb{R}$. $f(0) = 0$. $f(x,y) = xy(x^2 - y^2)/(x^2+y^2)$ if $(x,y) \neq 0$.

a). the existence of $D_1 f$ and $D_2 f$ at 0.

$D_1 f(0) = \lim_{t\to 0} (f(0+t, 0) - f(0))/t = \lim_{t\to 0} (t\cdot 0(t^2 - 0)/(t^2+0))/t = \lim_{t\to 0} 0 = 0$ exist

$D_2 f(0) = \lim_{s\to 0} (f(0, 0+s) - f(0))/s = \lim_{s\to 0} (0\cdot s(0 - s^2)/(0+s^2))/s = \lim_{s\to 0} 0 = 0$ exist.

b). $D_1 f(x,y) = \partial f(x,y)/\partial x = (x^2+y^2)^{-2}\left((x^2+y^2)(3x^2y - y^3) - xy(x^2-y^2)(2x)\right) = y(x^4 + 4x^2y^2 - y^4)/(x^2+y^2)^2$

$D_2 f(x,y) = \partial f(x,y)/\partial y = (x^2+y^2)^{-2}\left((x^2+y^2)(x^3 - 3xy^2) - xy(x^2-y^2)(2y)\right) = x(x^4 - 4x^2y^2 - y^4)/(x^2+y^2)^2$

where $(x,y) \neq 0$.

c) * Define $g_1(x,y) = (x^4 + 4x^2y^2 - y^4)/(x^2+y^2)^2$ then $D_1 f(x,y) = y\cdot g_1(x,y)$.

$g_1(x,y) = (x^4 + 4x^2y^2 - y^4)/(x^4 + 2x^2y^2 + y^4)$ ~~$= 1 + (2x^2y^2 - 2y^4)/(x^4+2x^2y^2+y^4)$~~.

but $x,y \in \mathbb{R} \Rightarrow x^2 \geq 0$ & $y^2 \geq 0 \Rightarrow -x^4 - 2x^2y^2 - y^4 \leq x^4 + 4x^2y^2 - y^4 \leq 2x^4 + 4x^2y^2 + 2y^4$

$\Rightarrow (-x^4 - 2x^2y^2 - y^4)/(x^2+y^2)^2 \leq g_1(x,y) \leq 2(x^2+y^2)^2/(x^2+y^2)^2$ also true when $(x,y) = 0$

$\Rightarrow -1 \leq g_1(x,y) \leq 2$. $g_1(x,y)$ is bdd, $D_1 f(x,y)$ = product of y & a bdd function.

~~Need to show $D_1 f(x,y)$ is ... $\lim_{h\to 0} D_1 f(x+h, y) - D_1 f(x,y) = \lim_{h\to 0}$~~

~~By def. $D_1 f(x_0, y) = \lim_{x\to x_0} \frac{f(x,y) - f(x_0,y)}{h}$ and $\lim_{h\to 0}$~~

~~$D_1 f = y(x^4+4x^2y^2-y^4)/(x^2+y^2)^2$ is cont. at $\mathbb{R} - \{0\}$ because it is a ratio of two cont. func~~

$y(x^4 + 4x^2y^2 - y^4)$ & $(x^2+y^2)^2$ is cont. on $\mathbb{R}^2 \Rightarrow D_1 f$ is cont. on $\mathbb{R}^2 - \{0\}$.

$\lim_{a\to 0} (D_1 f(a) - D_1 f(0)) = \lim_{a\to 0} y\, g_1(a) = 0$ because $\lim_{a\to 0} y = 0$ and $g_1(a)$ bdd by -1 & 2.

$\Rightarrow D_1 f$ is cont. at 0 $\Rightarrow D_1 f$ is cont. on $\mathbb{R}^2$ $\square$.

* Define $g_2(x,y) = (x^4 - 4x^2y^2 - y^4)/(x^2+y^2)^2$ then $D_2 f = x g_2$

$x,y \in \mathbb{R} \Rightarrow x^2 \geq 0$ & $y^2 \geq 0 \Rightarrow (x^4 + 2x^2y^2 + y^4)/(x^2+y^2)^2 \geq g_2(x,y) \geq -2(x^4 - 2x^2y^2 + y^4)/(x^2+y^2)^2 \Rightarrow g_2 \in [-2, 1]$.

$y(x^4 - 4x^2y^2 - y^4)$ & (x^2+y^2) cont. on $\mathbb{R}^2 \Rightarrow D_2 f$ is cont. on $\mathbb{R}^2 - \{0\}$.

$\lim_{a\to 0} (D_2 f(a) - D_2 f(0)) = \lim_{a\to 0} x g_2(a) = \lim_{a\to 0} x \lim_{a\to 0} g_2(a) = 0$ as $g_2(a)$ is bdd.

$\Rightarrow D_2 f$ is cont. at 0 $\Rightarrow D_2 f$ is cont. on $\mathbb{R}^2$. $\square$.

thus f is cont. diff. on $\mathbb{R}^2$. f is of C' on $\mathbb{R}^2$ $\square$.

d). ~~Def~~ $D_2D_1f(0) = \lim_{h\to0}\left(D_1f(0,h) - D_1f(0,0)\right)/h = \lim_{h\to0}\left(h(0^4+4 0^2h^2-h^4)/(0+h^2)^2\right)/h$
$= \lim_{h\to0} -\frac{h^4}{h^4} = -1$ exist.

$D_1D_2f(0) = \lim_{h\to0}\left(D_2f(h,0) - D_2f(0,0)\right)/h = \lim_{h\to0}\left(h(h^4-4 0^2h^2-0^4)/(h^2+0)^2\right)/h$
$= \lim_{h\to0} \frac{h^4}{h^4} = 1$ exist.

$D_2D_1f(0) = -1$ & $D_1D_2f(0) = 1$, exists but not equal.

Section 7

1. $f: \mathbb{R}^3 \to \mathbb{R}^2$. $f(0) = (1,2)$ $Df(0) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ $g: \mathbb{R}^2 \to \mathbb{R}^2$ $g(x,y) = (x+2y+1, 3xy)$.

Sol^n: $D(g\circ f)(0) = Dg(f(0)) \cdot Df(0)$.

10 $Dg = \begin{pmatrix} \partial g_1/\partial x & \partial g_1/\partial y \\ \partial g_2/\partial x & \partial g_2/\partial y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3y & 3x \end{pmatrix}$ $Dg(f(0)) = Dg(1,2) = \begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix}$

$\Rightarrow D(g\circ f)(0) = Dg(f(0)) \cdot Df(0) = \begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 \\ 6 & 12 & 21 \end{pmatrix}$ □.

2. $f: \mathbb{R}^2 \to \mathbb{R}^3$ $g: \mathbb{R}^3 \to \mathbb{R}^2$ $f(x) = (e^{2x_1+x_2}, 3x_2 - \cos x_1, x_1^2+x_2+2)$. $g(y) = (3y_1+2y_2+y_3^2, y_1^2 - y_3+1)$.

a). $F(x) = g(f(x))$. Find $DF(0)$.

$f(0) = (1,-1,2)$ $Df(x) = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 \end{pmatrix} = \begin{pmatrix} 2e^{2x_1+x_2} & e^{2x_1+x_2} \\ \sin x_1 & 3 \\ 2x_1 & 1 \end{pmatrix}$ $Df(0) = \begin{pmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 1 \end{pmatrix}$

$g(f(0)) = (5,0)$ $Dg(y) = \begin{pmatrix} \partial g_1/\partial y_1 & \partial g_1/\partial y_2 & \partial g_1/\partial y_3 \\ \partial g_2/\partial y_1 & \partial g_2/\partial y_2 & \partial g_2/\partial y_3 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2y_3 \\ 2y_1 & 0 & -1 \end{pmatrix}$ $Dg(f(0)) = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & -1 \end{pmatrix}$

$\Rightarrow DF(0) = D(g\circ f)(0) = Dg(f(0)) \cdot Df(0) = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 13 \\ 4 & 1 \end{pmatrix}$ □

b). $G(y) = f(g(y))$. Find $DG(0)$.

$g(0) = (0,1)$ $Dg(0) = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $f(g(0)) = f(0,1)$ ~~e~~ $= (e, \cancel{3}2, 3)$ $Df(g(0)) = \begin{pmatrix} 2e & e \\ 0 & 3 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow DG(0) = D(f\circ g)(0) = Df(g(0)) \cdot Dg(0) = \begin{pmatrix} 2e & e \\ 0 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 6e & 4e & -e \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{pmatrix}$ □

3. $f:\mathbb{R}^3 \to \mathbb{R}$ $\quad g:\mathbb{R}^2 \to \mathbb{R}$ differentiable. $\quad F:\mathbb{R}^2 \to \mathbb{R}$ $\quad F(x,y) = f(x,y,g(x,y))$.

a). $DF = (\partial F/\partial x \quad \partial F/\partial y) = (df(x,y,g(x,y))/dx \quad df(x,y,g(x,y))/dy)$

$$= \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial g}\frac{\partial g}{\partial x} \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial g}\frac{\partial g}{\partial y}\right)$$

$$\cancel{= D_1 f + D_3 f}$$

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$$DF(x,y) = (D_1 f(x,y,g(x,y)) + D_3 f(x,y,g(x,y))\, D_1 g(x,y) \quad D_2 f(x,y,g(x,y)) + D_3 f(x,y,g(x,y))\, D_2 g(x,y)).$$

$$DF = (D_1 f + D_3 f \cdot D_1 g \quad D_2 f + D_3 f \cdot D_2 g). \ \square.$$

b). $F(x,y) = 0 \quad \forall (x,y)$ $\qquad$ Find $D_1 g$ & $D_2 g$.

~~$F(x,y)=0 \Rightarrow DF=0$ (because F is constant function)~~

~~$\Rightarrow D_1 g = -D_1 f/D_3 f$~~

~~$F(x,y)=0 \Rightarrow f(x,y,g(x,y))=0$~~

$$F(x,y)=0 \Rightarrow DF=0 \Rightarrow \begin{cases} D_1 f + D_3 f \cdot D_1 g = 0 \\ D_2 f + D_3 f \cdot D_2 g = 0 \end{cases} \Rightarrow \begin{cases} D_1 g = \begin{cases} -D_1 f/D_3 f & \text{if } D_3 f \neq 0. \\ \text{any functions from } \mathbb{R} \text{ to } \mathbb{R} & \text{if } D_3 f = 0. \end{cases} \\ D_2 g = \begin{cases} -D_2 f/D_3 f & \text{if } D_3 f \neq 0. \\ \text{any functions from } \mathbb{R} \text{ to } \mathbb{R} & \text{if } D_3 f = 0. \end{cases} \end{cases} \quad \square.$$