Problem Set 19 — MAT257

April 5, 2017

Disclaimer—This page has been typeset by a student as a *convenient* consolidation of the homework problems. There inevitably will be mistakes; always defer to the official handout!

---

1 Munkres §37 (pp.308–309)

4. Let M be the 2-manifold in $\mathbb{R}^3$ consisting of all points $\mathbf{x}$ such that

$$4(x_1)^2 + (x_2)^2 + 4(x_3)^2 = 4 \quad \text{and} \quad x_2 \geq 0.$$

Then ∂M is the circle consisting of all points such that

$$(x_1)^2 + (x_3)^2 = 1 \quad \text{and} \quad x_2 \geq 0.$$

See Figure 37.5 (Munkres, p.309). The map

$$\alpha(u, v) = (u, 2\sqrt{1 - u^2 - v^2}, v),$$

for $u^2 + v^2 < 1$, is a coordinate patch on M that covers $M \setminus \partial M$. Orient M so that α belongs to the orientation, and give ∂M the induced orientation.

(a) What normal vector corresponds to the orientation of M? What tangent vector corresponds to the induced orientation of ∂M?

(b) Let ω be the 1-form $\omega = x_2\, dx_1 + 3x_1\, dx_3$. Evaluate $\int_{\partial M} \omega$ directly.

(c) Evaluate $\int_M d\omega$ directly, by expressing it as an integral over the unit disc in the (u, v) plane.

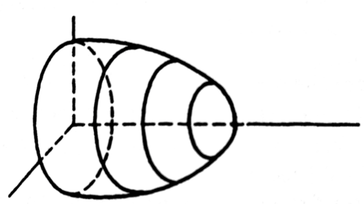

Figure 37.5

5. The 3-ball $B^3(r)$ is a 3-manifold in $\mathbb{R}^3$; orient it naturally and give $S^2(r)$ the induced orientation. Assume that ω is a 2-form defined in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that

$$\int_{S^2(r)} \omega = a + \frac{b}{r}$$

for each $r > 0$.

(a) Given $0 < c < d$, let M be the 3-manifold in $\mathbb{R}^3$ consisting of all $\mathbf{x}$ with $c \leq \|\mathbf{x}\| \leq d$, oriented naturally. Evaluate $\int_M d\omega$.

(b) If $d\omega = 0$, what can you say about a and b?

(c) If $\omega = d\eta$ for some η in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$, what can you say about a and b?

6. Let M be an oriented $(k+l+1)$-manifold without boundary in $\mathbb{R}^n$. Let ω be a k-form and let η be an l-form, both defined in an open set of $\mathbb{R}^n$ about M. Show that

$$\int_M \omega \wedge d\eta = a \int_M d\omega \wedge \eta$$

for some a, and determine a. (Assume M is compact.)

2 Munkres §38 (pp.320–322)

2. Let G be a vector field defined in $A = \mathbb{R}^n \setminus \{\mathbf{0}\}$ with $\operatorname{div} G = 0$ in A.

(a) Let M_1 and M_2 be compact n-manifolds in $\mathbb{R}^n$, such that the origin is contained in both $M_1 \setminus \partial M_1$ and $M_2 \setminus \partial M_2$. Let N_i be the unit outward normal vector field to ∂M_i, for $i = 1, 2$. Show that

$$\int_{\partial M_1} \langle G, N_1 \rangle \, \mathrm{d}V = \int_{\partial M_2} \langle G, N_2 \rangle \, \mathrm{d}V.$$

[*Hint:* Consider for the case where $M_2 = B^n(\epsilon)$ and is contained in $M_1 \setminus \partial M_1$. See Figure 38.2 (Munkres, p.321).]

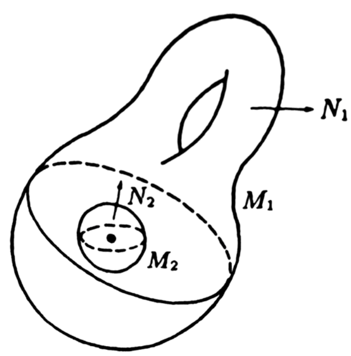

Figure 38.2

(b) Show that as M ranges over all compact n-manifolds in $\mathbb{R}^n$ for which the origin is not in ∂M, the integral

$$\int_{\partial M} \langle G, N \rangle \, \mathrm{d}V,$$

where N is the unit normal to ∂M pointing outwards from M, has only two possible values.

3. Let G be a vector field in $B = \mathbb{R}^n \setminus \{\mathbf{p}, \mathbf{q}\}$ with $\operatorname{div} G = 0$ in B. As M ranges over all compact n-manifolds in $\mathbb{R}^n$ for which $\{\mathbf{p}\}$ and $\{\mathbf{q}\}$ are not in ∂M, how many possible values does the integral

$$\int_{\partial M} \langle G, N \rangle \, \mathrm{d}V$$

have? (Here N is the unit normal to ∂M pointing outwards from M.)

5. Let S be the subset of $\mathbb{R}^3$ consisting of the union of:

i the z-axis,

ii the unit circle $x^2 + y^2 = 1$, $z = 0$,

iii the points $(0, y, 0)$ with $y \geq 1$.

Let A be the open set $\mathbb{R}^3 \setminus S$ of $\mathbb{R}^3$. Let C_1, C_2, D_1, D_2, D_3 be the oriented 1-manifolds in A that are pictured in Figure 38.3 (Munkres, p.322). Suppose that F is a vector field in A, with $\operatorname{curl} F = \mathbf{0}$ in A, and that

$$\int_{C_1} \langle F, T\rangle \, ds = 3 \quad \text{and} \quad \int_{C_2} \langle F, T\rangle \, ds = 7.$$

What can you say about the integral

$$\int_{D_i} \langle F, T\rangle \, ds$$

for $i = 1, 2, 3$? Justify your answers.

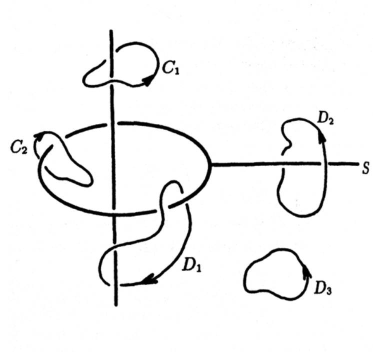

Figure 38.3

3 "Ponder…"

Challenge! Why does a planimeter work? See images at
http://drorbn.net/AcademicPensieve/Classes/1617-257b-AnalysisII/Planimeter/