MAT257 - 28/09/2016 tutorial

September 28, 2016

Unless otherwise stated, X and Y are metric spaces.

Exercise 1: Continuity and closed sets

Prove the following characterization of continuity: a map $f : X \to Y$ is continuous if and only if for any subset $A \subset Y$, one has $\overline{f^{-1}(A)} \subset f^{-1}(\overline{A})$.

Exercise 2: Local continuity

a) Recall that a map between topological spaces $f : X \to Y$ is continuous at a point $x \in X$ when for any open neighborhood $\mathcal{V}_{f(x)} \subset Y$, there exists a neighborhood $\mathcal{U}_x \subset X$ of x such that $f(\mathcal{U}_x) \subset \mathcal{V}_{f(x)}$. Prove that $f : X \to Y$ is continuous if and only if f is continuous at every $x \in X$.
b) Consider the following property for a map $f : X \to Y$:

$$\text{(P) } \forall x \in X, \exists \mathcal{U}_x \subset X \text{ open nhd such that } f_{|\mathcal{U}_x} \text{ is continous.}$$

Show that $f : X \to Y$ is continuous if and only if it satisfies property (P).

Exercise 3: Random facts on compactness

Prove the following results:
a) A continuous image of a compact set is compact.
b) A closed subset of a compact set is compact.
c) In general, a topological space is said to be Hausdorff if for any two of its points, there exist disjoint open neighborhoods. Metric spaces are always Hausdorff, while locally convex spaces and algebraic varieties are not. Prove that in a Hausdorff space, any compact subset is closed.

Exercise 4: Sequential compactness

Generally speaking, a subset K of a topological space X is called sequentially compact if any sequence $\{x_n\}_{n\in\mathbb{N}} \subset K$ admits a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}} \subset K$. Prove that in the case of metric spaces, the notions of compactness and sequential compactness are equivalent.

Exercise 5: Compact vs closed and bounded

Even in the case of a metric space, a closed and bounded subset need not be compact. Illustrate this fact with an example.