MAT240 - Tutorial    Nov. 2, 2006.

ex  A, P  n×n matrices, $A^T A = I$.

Show $(APA^T)^2 = AP^2A^T$ by using index notation.

$(AP)_{ij} = \sum_{k=1}^{n} A_{ik} P_{kj}$

$(A^T)_{ij} = A_{ji}$

$(APA^T)_{ij} = \sum_{k=1}^{n} \sum_{l=1}^{n} A_{ik} P_{kl} A^T_{lj} \rightarrow A_{jl}$

$(APA^T)^2 = \sum_{k,l} A_{ik} P_{kl} \underbrace{A_{cl} A_{ca}}_{\delta_{la}} P_{ab} A_{jb}$

$l = a, b, c$

Rule

ex $= (AP^2A^T)_{ij}$

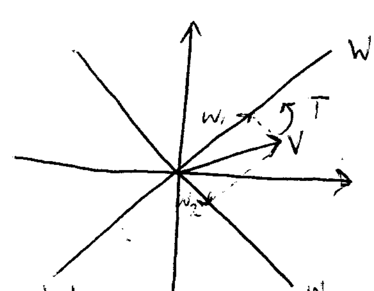
Every term is equal to 0 except when $l = a$, so can set $a = l$ and cancel $\delta_{ll}$.

$(A^TA)_{ij} = I_{ij} = \delta_{ij}$

$= \sum_k (A^T)_{ik} A_{kj}$

$= \sum_k A_{ki} A_{kj}$

6.24  ex  $V = W_1 \oplus W_2$
P.76

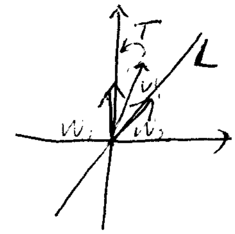



$v = w_1 + w_2$ uniquely.

Projection on $W_1$ along $W_2$
$T(v) = w_1$

$V = \mathbb{R}^2$

24.a) $T =$ proj on y-axis along x-axis.



$T(a,b) = (0, b)$ → vector

b) $T =$ ———— $L = \{(s,s) \mid s \in \mathbb{R}\}$

$T(a,b) = (0, b-a)$

y-axis

$v = (a,b) = (\alpha, \alpha) + (\beta, \beta) = (0, b-a) + (a, a)$

$a = \beta$

$b = \alpha + \beta$

$\alpha = b - a$

$\beta = a$

26.a) T is linear $\quad T(v_1 + v_2) = T(v_1) + T(v_2)$

Exercise.

$W_1 \overset{?}{=} \{x \in V \mid T(x) = x\}$

$T(x) = x \Rightarrow x \in W_1$

$x = w_1 + w_2$ uniquely.

$w_1 = w_1 + w_2 \Rightarrow w_2 = 0 \Rightarrow x = w_1 \in W_1$

b) ① Range(T) $= W_1$ ($R(T)$) ② Kernel(T) $= W_2$ ($N(T)$)

① $R(T) \subseteq W_1$ by def'n

$W_1 \subseteq R(T) \because T(w_1) = w_1 \ \forall w_1 \in W_1$

② $N(T) \supseteq W_2 \quad \because 0 = T(w_2 = 0 + w_2)$

$N(T) \subseteq W_2$

$T(w_1 + w_2) = 0$

$\quad \| $

$\quad w_1$

$\Rightarrow x = w_2 \in W_2$

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c) $W_i = V \quad T = I$

d) $W_i = \{0\} \quad T = 0$

$T: V \to V$ linear
W subspace
W is T invariant if $T(W) \subseteq W$

28. $\{0\}$, $V$, ① $R(T)$, ② $N(T)$ are T invariant.

① $T(R(T)) \overset{?}{\subseteq} R(T)$ ✓ def'n of range

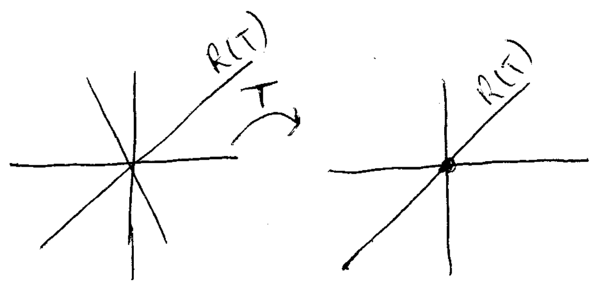
② $T(N(T)) \in \{0\} \subseteq N(T)$

$T_W$ is the restriction of T to W. $T_W: W \to W$.

29. Show $T_W$ is linear. □

30. T = proj on W along W'
Show W is T invariant.
$T(w) = w \quad \forall w \in W$
$T_W = I$

31. $V = R(T) \oplus W$
W is T invariant.




a) $W \subseteq N(T)$

$T(W) \subseteq W \cap R(T) = \{0\}$

↑ def'n T invariant & R(T)   ↑ ∵ $R(T) \oplus W = V$

$\Rightarrow W \subseteq N(T)$

31. b) $\dim V < \infty \Rightarrow W = N(T)$

$\dim R(T) + \dim W = \dim V = \dim R(T) + \dim N(T)$
$\Rightarrow W = N(T)$ by Dim Thm. & Q5 on test

c) $\dim V = \infty$, counterexample.
$W \neq N(T)$ $N(T) = \{\text{constants}\}$
$V = P(\mathbb{R})$ $W = \{0\}$.
$T = \frac{d}{dx}$ Show $V \stackrel{?}{=} R(T)$

Given $p \in P(\mathbb{R})$ let $q = \int_0^x p(y)\,dy$

$\frac{d}{dx} q = p$.

$\therefore P(\mathbb{R}) = R(T)$.
$p = T(q)$

21. $V = \{(a_1, a_2, \ldots) \mid a_i \in \mathbb{R}\}$

$T(a_1, a_2, \ldots) = (a_2, a_3, \ldots)$
$U(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$

b) $U, T : V \to V$
T is onto, not 1-1.
$\because T(0, a_2, a_3, \ldots) = (a_2, a_3, \ldots)$
not 1-1.
$\because T(1, 0, \ldots) = (0, 0, \ldots)$

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4. $T: M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$

$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + 2dx + bx^2$

$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$\gamma = \{1, x, x^2\}$

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

10. $$[T]_\beta = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \vdots \\ & 0 & 1 & \ddots & \vdots \\ \vdots & & 0 & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

11. $T: V \to V$ $\dim V = n$. $W$ $T$ invariant $\dim W = k$

** Find basis $\beta$ s.t.

$$[T]_\beta = \begin{bmatrix} \overbrace{A}^{k\times k} & B \\ 0 & \underbrace{C}_{(n-k)(n-k)} \end{bmatrix}$$

let $v_1, \ldots, v_k$ be a basis of $W$.
expand to $v_1, \ldots, v_n$ of $V$

$$[T]_\beta = \left( [T(v_1)]_\beta \,\middle|\, [T(v_2)]_\beta \,\middle|\, \cdots \,\middle|\, [T(v_k)]_\beta \,\middle|\, * \right).$$

$T(v_i) \in W$ $\forall i = 1, \ldots, k$.
i.e. $T(v_i) \in \operatorname{span}\{v_1, \ldots, v_k\}$.
$\Rightarrow$ the $v_{k+1}, \ldots, v_n$ components of $T(v_1), \ldots, T(v_k)$ are 0.

$\therefore$ 0 block is 0

2.2

12. $\dim V < \infty$

$T = $ proj on $W$ along $W'$

Find $\beta$ s.t. $[T]_\beta$ is diagonal matrix.

Pick a basis $\{v_1, \dots, v_k\}$ for $W$. ~~extend to $\{v_1, \dots, v_n\}$ for $V$.~~ Pick a basis $\{v_{k+1}, \dots, v_n\}$ for $W'$

$$[T]_\beta = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \{v_1, \dots, v_n\}$ is a basis for $V$

$\because W \oplus W' = V$

13. $T, U \neq 0 : V \to W$ $\quad R(T) \cap R(U) = \{0\}$.

Show $\{U, T\}$ is lin indep in $\mathcal{L}(V, W)$.

$aU + bT = 0$

$aU = -bT$

$\forall x \in V$

$\Rightarrow aU(x) = -bT(x)$

$\Rightarrow aU(x) \in R(T) \cap R(U) = \{0\}$

$\Rightarrow a = 0 \Rightarrow b = 0$.

14. $V = P(\mathbb{R})$

$T_j : V \to V$

$T_j(f) = f^{(j)}$

$\{T_1, \dots, T_n\}$ is lin indep $\forall n > 0$.

$a_1 T_1 + a_2 T_2 + \dots + a_n T_n = 0$

$a_1 T_1(x) + a_2 T_2(x) + \dots + a_n T_n(x) = 0$.

$a_1T_1(x) = a_1$ $\quad a_1 = 0$

next, $a_2 T_2(x^2) + \dots + a_n T_n(x^2) = 0$

$a_2T_2(x^2) = 2a^2$

$a_2 = 0$

$\vdots$ Induction. $\square$.