Feb. 8^{th} Wed. Hour 051

Today: move on alternating tensors

Read along: Section 27-29

S_n: {bijections $\sigma: \underline{n} \circlearrowleft$ }

$\exists!$ Sign : $S_n \to \{\pm\}$ Such that $(-1)^{\sigma\cdot\tau} = (-1)^{\sigma}\cdot(-1)^{\tau}$, and $(-1)^{ij} = -1$

Thm: $\forall I \in \binom{n}{k}$, $\exists!$ $\psi_I \in A^k(V)$, such that $\psi_I(a_J) = \delta_{IJ}$.

Those $\{\psi_I\}$ make a basis of A^k_J, hense $\dim A^k(n) = \binom{n}{k}$

$F \in A^k(V)$, $F^{\sigma}(X_1,\cdots,X_k) = F(X_{\sigma 1},\cdots,X_{\sigma k})$. $F^{\sigma} = (-1)^{\sigma}\cdot F$,

$\binom{X}{k}$ = Subsets of size k of X, $\binom{n}{k} = (i_1, i_2, \cdots, i_k)$ where $1 \leq i_{\alpha-1} < i_\alpha < i_{\alpha+1} \leq n$

Def. $\psi_I = \sum_{\sigma\in S_k} (-1)^{\sigma} \phi_I^{\sigma} \in \mathcal{L}^k(V)$, Claim: ψ_I is in $A^k(V)$

$$(\psi_I)^{\tau} = \left(\sum_{\sigma\in S_k} (-1)^{\sigma} \phi_I^{\sigma}\right)^{\tau} \quad (*)$$

Aside: $f, g \in \mathcal{L}^k$, $\sigma \in S_k$, $(af+bg)^{\sigma} = af^{\sigma} + bg^{\sigma}$

$$(*) = \sum_{\sigma\in S_k} (-1)^{\sigma} (\phi_I^{\sigma})^{\tau} = \sum_{\sigma\in S_k} (-1)^{\overset{\tau^{-1}\cdot\lambda}{\sigma}} \phi_I^{\overset{\lambda}{\tau\cdot\sigma}}$$

letting $\lambda = \tau\cdot\sigma$

$\tau^{-1}(\lambda) = \sigma$

Aside: $((F^{\sigma})^{\tau} = F^{\tau\cdot\sigma} \Rightarrow$ $= \sum_{\lambda\in S_k} (-1)^{\tau^{-1}\lambda} \phi_I^{\lambda}$

$\hookrightarrow (-1)^{\tau^{-1}}\cdot(-1)^{\lambda} = (-1)^{\tau}\cdot(-1)^{\lambda}$

$$= (-1)^{\tau}\cdot\sum_{\lambda\in S_k} (-1)^{\lambda}\phi_I^{\lambda} = (-1)^{\tau}\psi_I$$

$F(X,Y) \rightsquigarrow F(X,Y) - F(Y,X) = g(X,Y)$

$g(Y,X) = F(Y,X) - F(X,Y) = -g(X,Y)$

$F(X,Y,Z) \rightsquigarrow F(X,Y,Z) - F(Y,X,Z)$

$- F(X,Z,Y) - F(Z,Y,X)$

$+ F(Y,Z,X) + F(Z,X,Y)$

extend to k, then check

$\psi_I(a_J)$, where $I, J \in \binom{n}{k}$, $I = (i_1 < \cdots < i_k)$, $J = (j_1 < \cdots < j_k)$

$= \sum_{\sigma\in S_k} (-1)^{\sigma} \phi_I^{\sigma}(a_{j_1},\cdots,a_{j_k}) = \phi_I(a_J) = \delta_{IJ}$

$+ \sum \pm \phi_I$ (lists of vectors not in assending order) $= 0$

In general, $\phi_I(a_k) = \delta_{Ik}$

Still need uniqueness (trivial)
If ψ_I is alternating, it's determined by its vectors on ascending sequences of basis vectors. $\Rightarrow \psi_I(a_J)$, $J \in \binom{n}{k}$, determines ψ_I.
Claim $\{\psi_I\}$ is linearly independent in $A^k(V)$

Pf: Assume $\sum_{I \in \binom{n}{k}} \alpha_I \psi_I = 0$, each $\alpha_i \in \mathbb{R}$, evaluate on a_J, both sides

where $J \in \binom{n}{k}$. $\sum_I \alpha_I \psi_I(a_J) = \alpha_J = 0$. $\square$.

Example: $U = \mathbb{R}^3 (a_1, a_2, a_3) = (e_1, e_2, e_3)$, $\phi_I(a_J) = \delta_{IJ}$

$$\phi_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \phi_1(x_1 e_1, x_2 e_2, x_3 e_3) \Rightarrow x_1$$

$$\phi_2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2, \quad \phi_3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3.$$

$I = (1\ 2)$, $n=3$, $k=2$

$$\phi_I\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = \phi_1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \phi_2\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 \cdot y_2$$

$$\phi_{(21)}(\vec{x}, \vec{y}) = x_2 \cdot y_1.$$

$$\psi_{(12)}(\vec{x}, \vec{y}) = \phi_{(12)}(\vec{x}, \vec{y}) - \phi_{(21)}(\vec{x}, \vec{y})$$

$$= x_1 y_2 - x_2 y_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

next time: $\psi_I(\cdots k \text{ vectors}) = \det(k \times k)$