Def: $S \subset V$ is called "linearly dependend" if you can find $z_1, \dots z_n \in S$ different from each other & $c_1 \dots c_n \in F$ not all of $(c_1, \dots c_n)$ which are 0, so that $\sum c_i z_i = 0$, otherwise S is called linearly independent.

Example: In $\mathbb{R}^3$ $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$ is lin. dep.

$$1 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (-2) \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = 0$$

Ex2: $\mathbb{R}^n$, $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$ i'th row $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $S = \{e_1, e_2, e_3, \dots e_n\}$

Claim: S is lin. independent.

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0 = \sum_{i=1}^{n} a_i e_i = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \Rightarrow \begin{matrix} a_1 = 0 \\ a_2 = 0 \\ \vdots \\ a_n = 0 \end{matrix}$$

It is not the case not all a_i's are 0. $\Rightarrow$ not linearly dependent.

Claim: $S \subset V$ is lin. indep. iff whenever $\sum a_i u_i = 0$ then $\forall i\ a_i = 0$
& $u_i \in S$ (u_i's are distinct)

Comments: 1. $\emptyset$ = the empty set $\subset V$ is lin. independent

2. Suppose $u \in V$ $\{u\}$ is linearly independant $\Leftrightarrow u \neq 0$
"singleton set"

$\{0\}$ lin. dep $7 \cdot 0 = 0$

$u \neq 0$

$a.u = 0$ & $a \neq 0$

$\Rightarrow a^{-1} a u = 0$

$u = 0$ ↯ So no such a exists. So $\{u\}$ is not lin. dependent

3. If $S_1 \subset S_2 \subset V$

a. If S_1 is lin. dependent so is S_2

b. If S_2 is lin. indep. so is S_1

c. If S_1 generates V so does S_2

d. If S_2 doesn't generate, neither does S_1

4. If S is linearly indep. in V and $v \notin S$

then $S \cup \{v\}$ is lin. dep. iff $v \in \operatorname{span} S$

<u>prf:</u> $\Leftarrow$: Assume $v \in \operatorname{span} S$

$v = \sum a_i u_i$ where $u_i \in S$

$\sum a_i u_i - 1v = 0$ this is a lin. comb. of elements in $S \cup \{v\}$

in which not all coeff's are 0 and which adds to 0. So $S \cup \{v\}$ is lin. dep.

$\Rightarrow$: Assume

$\Rightarrow$: Assume $S \cup \{v\}$ is lin. dep $\Rightarrow$ can find a lin. comb.

$$(*) \quad \sum a_i u_i + bV = 0$$

where $u_i \in S$ not all of a_i's & b are 0.

If $b = 0$, $\sum a_i u_i = 0$ & not all a_i's are 0.

$\Rightarrow S$ is lin. dep. $\lightning$, so $b \neq 0$

so divide by b: $*$ becomes

$$\sum \frac{a_i}{b} u_i + V = 0 \Rightarrow V = \sum \frac{-a_i}{b} u_i \Rightarrow V \in \operatorname{Span} S. \quad \blacksquare$$

Def: A basis of a v.s. V is a subset ~~is a subset~~ $\beta \subset V$ s.t.

① β generates V ($V = \operatorname{span} \beta$)

② β is lin. indep

Examples.

① $\beta = \emptyset$ is a basis of $\{0\}$

② V be $\mathbb{R}$ as a vector space over $\mathbb{R}$ $\beta = \{5\}$, $\beta = \{1\}$ are bases

③ V be $\mathbb{C}$ as a v.s. over $\mathbb{R}$

$\beta = \{1, i\}$ 1. $z = a.bi = a \cdot 1 + b \cdot i$ so $\{1, i\}$ generate

2. show $1, i$ are lin. indep.

assume $a \cdot 1 + bi = 0$ where $a, b \in \mathbb{R}$

$\Rightarrow a + bi = 0$

$\Rightarrow a = 0,\ b = 0$ $\therefore$ 1 and i are lin. indep. $\boxtimes$

4. $V = \mathbb{R}^n = \left\{ \begin{pmatrix} \cdot \\ \vdots \\ \cdot \end{pmatrix} \right\}$ $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$e_1 \cdots e_n$ are a basis of V

① they span $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum a_i e_i$

② $\sum a_i e_i = 0 = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = 0 \Rightarrow a_i = 0 \ \forall i$

5. In $V = P_3(\mathbb{R})$ $\beta = \{1, x, x^2, x^3\}$ $2x^3 - 7 = 2(x^3) - 7(1)$

6. In $P_1(\mathbb{R}) = \{ax + b\}$ $\beta = \{\underset{u_1}{1+x}, \underset{u_2}{1-x}\}$ is a basis.

① $\frac{1}{2}(u_1 + u_2) = \frac{1}{2} \cdot 2 = 1$ So $1, x \in \operatorname{span} \beta$

$\frac{1}{2}(u_1 - u_2) = \frac{1}{2} 2x = x$ so $\underset{P_1\{\mathbb{R}\}}{\underset{\parallel}{\operatorname{span}\{1, x\}}} \subset \operatorname{span} \beta$

② Assume

$a u_1 + b u_2 = 0 \Rightarrow a(1+x) + b(1-x) = 0$

$(a+b) + (a-b)x = 0 \Rightarrow \begin{matrix} a+b=0 \\ a-b=0 \end{matrix} \therefore \begin{matrix} a=0 \\ b=0 \end{matrix}$