19/09/06

06-240

F1 : Commutativity
F2 Associativity
F3 Units
F4 Inverses
F5 Distributivity

$\mathbb{C} := \{(a,b) : a, b \in \mathbb{R}\}$
$0_{\mathbb{C}} := (0,0) \qquad 1_{\mathbb{C}} = (1,0)$

$(a,b)+(c,d) := (a+c, b+d)$
$(a,b)\cdot(c,d) := (ac-bd, ad+bc)$

F1,2,3,5 are mechanical F4: $-(a,b) = (-a,-b)$

$$(a,b)^{-1} = \left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right)$$

$$(a,b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) \underset{\text{def}}{\overset{\text{by}}{=}} \left(a\cdot\frac{a}{a^2+b^2} - b\,\frac{-b}{a^2+b^2},\ a\,\frac{-b}{a^2+b^2} + b\,\frac{a}{a^2+b^2}\right)$$

$$= \left(\frac{a.a - b(-b)}{a^2+b^2}, \frac{a(-b)+ba}{a^2+b^2}\right) = (1,0) = 1_{\mathbb{C}}$$

$$a^2+b^2=0 \iff a=0 \text{ and } b=0 \iff (a,b)=(0,0)$$

Notation:
$(0,1) =: i \qquad (a,0) =: a$

$$(0,1)^2 = (0,1)(0,1) = (0-1.1, 0.1+1.0) = (-1,0) = -(1,0) = -1_{\mathbb{C}}$$

Now $(a,b) = a+bi$

Indeed $(a,0)+(b,0)\cdot(0,1) = (a,0)+(0,b) = (a,b)$ □

Def: If $z = a+bi$ $(= (a,b))$ is a complex number,

define $\bar{z}$ = "the conjugate of $\bar{z}$" $= a - bi$ $(= (a,-b))$

Example: $\overline{-157+240i} = -157-240i$

$\overline{-157-240i} = -157+240i$

Thm: $\bar{\bar{z}} = z$

Easy Claims:

1. $\overline{z+w} = \bar{z} + \bar{w}$
2. $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$
3. $\overline{-z} = -\bar{z}$
4. $z \neq 0$, $\overline{z^{-1}} = \bar{z}^{-1}$

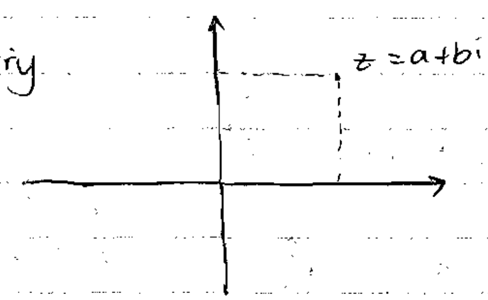
to prove set $z = a+bi$, $w = c+di$ and follow definitions

$$z \cdot \bar{z} = (a+bi)(a-bi) = a^2 - (bi)^2 = a^2+b^2 =: |z|^2$$

if $z = a+bi$

i.e. define $|z| = \sqrt{a^2+b^2} = \sqrt{z \cdot \bar{z}}$

1. Geometry



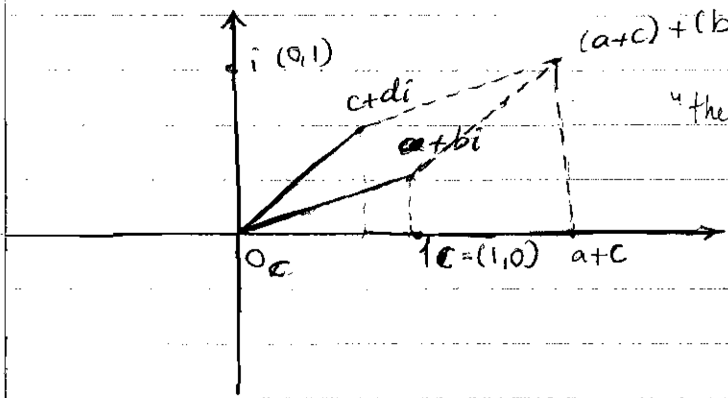
Complex number is a point in geometry.

remember this

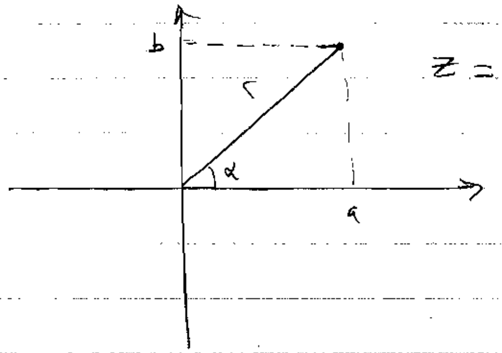
2. Algebra

$$z\bar{z} = |z|^2 \xrightarrow[|z|^2 \neq 0,\ z=0]{\text{if}} z \cdot \underbrace{\frac{\bar{z}}{|z|^2}}_{z^{-1}} = 1 \Rightarrow z^{-1} = \frac{\bar{z}}{|z|^2}$$

not $(a,b)^{-1} = \frac{a}{a^2+b^2}, -\frac{b}{a^2+}$



"the parallelogram law"



$z = r \cdot e^{i\alpha} = (r\cos\alpha, r\sin\alpha)$

the polar coordinates of z

$\cos\alpha = \frac{a}{r} \therefore a = r\cos\alpha$

$\sin\alpha = \frac{b}{r} \therefore b = r\sin\alpha$

$z_1 = r_1 e^{i\alpha_1} \quad z_2 = r_2 e^{i\alpha_2}$

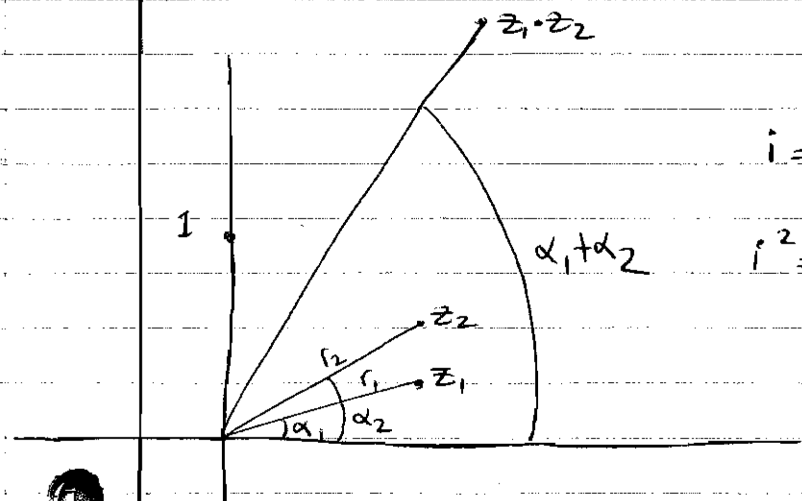
$z_1 \cdot z_2 = (r_1\cos\alpha_1 + ir_1\sin\alpha_1)(r_2\cos\alpha_2 + ir_2\sin\alpha_2)$

$= r_1r_2\cos\alpha_1\cos\alpha_2 - r_1r_2\sin\alpha_1\sin\alpha_2 + i(r_1r_2\cos\alpha_1\sin\alpha_2 + r_1r_2\sin\alpha_1\cos\alpha_2)$

$= r_1r_2(\cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2) + ir_1r_2(\cos\alpha_1\sin\alpha_2 + \sin\alpha_1\cos\alpha_2)$

$$= \underbrace{r_1 r_2}_{R} \cos \underbrace{(\alpha_1 + \alpha_2)}_{\beta} + i \underbrace{r_1 r_2}_{R} \sin \underbrace{(\alpha_1 + \alpha_2)}_{\beta}$$

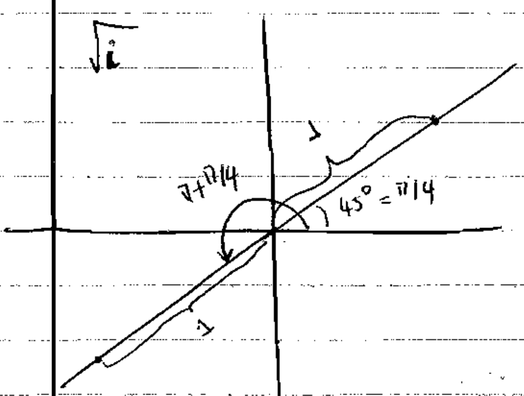
$$= R\cos\beta + iR\sin\beta \doteq Re^{i\beta} = (r_1 r_2) e^{i(\alpha_1 + \alpha_2)}$$



$$i = 1 \cdot e^{i \cdot \pi/2}$$

$$i^2 = 1 \cdot 1 \cdot e^{i(\pi/2 + \pi/2)} = 1 \cdot e^{i\pi}$$

$\sqrt{i}$



$$\sqrt{i} = 1 \cdot e^{i\frac{\pi}{4}} = 1 \cdot \cos\frac{\pi}{4} + 1 \cdot i \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

or

$$\sqrt{i} = 1 \cdot e^{i\left(\pi + \frac{\pi}{4}\right)}$$

$\left(\sqrt{-1} = \begin{matrix} i & (0,1) \\ \text{or} & \\ -i & (0,-1) \end{matrix} \right.$

we simply chose i)

Look the other way

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{(-1)}\sqrt{(-1)} = i \cdot i = -1$$

Divide 240 by 7, what's the reminder?

$240 = 238 + 2 = 34 \cdot 7 + \underbrace{2}_{\text{remainder}}$

Claim: Let $n > 1$ & x be integers. Then there are unique integer q and r s.t. $x = q \cdot n + r$ and $0 \leq r \leq n-1$

Def: In this case $r =: x \bmod n$

$x \text{ rem } n$

Example $240 \text{ rem } 7 = 2$

$-240 \text{ rem } 7 = 5$

Definition: $\mathbb{Z} = \{0, 1, 2, \dots (n-1)\}$

$\mathbb{Z} = \{0, 1\}$

$0_{\mathbb{Z}/n} = 0 \qquad 1_{\mathbb{Z}/n} = 1$

we take the reminders then add them since we don't want to pass $n-1$ (i.e $r_1 = n-2$, $r_2 = n-1$, $r_1 + r_2 = 2n-3$ may pass $n-1$.)

If $r_1 \& r_2 \in \mathbb{Z}/n$ $\quad r_1 +_{\mathbb{Z}_n} r_2 = (r_1 + r_2) \text{ rem} n$

$r_1 \times_{\mathbb{Z}} r_2 := (r_1 \cdot r_2) \text{ rem } n$

Example:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

X	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Theorem: 1. $\mathbb{Z}/n$ with the operations just defined satisfies F1, F2, F3, F4 for addition at least, F5

2. If n is a prime s.t $n = 5, 7$ then F4 for multiplication ~~n=~~ hold too so $\mathbb{Z}/n$ is a field

Example: in $\mathbb{Z}/5$

$-1 = 4$ $\quad 1^{-1} = 1$

$-2 = 3$ $\quad 2^{-1} = 3 \quad (2 \cdot 3 = 1)$

$-3 = 2$ $\quad 3^{-1} = 2$

$-4 = 1$ $\quad 4^{-1} = 4$

$\cancel{0^{-1}}$ not expected anyways

In $\mathbb{Z}/4$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

X	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	3	2

There is no 2^{-1} in $\mathbb{Z}/4$

$\therefore$ $\mathbb{Z}/4$ is not a field