Then suppose that:

$$0 = \alpha_1 x + \alpha_2(x^2+1) + \alpha_3(x^3+2x)$$
$$= \alpha_3 x^3 + \alpha_2 x^2 + (2\alpha_3 + \alpha_1)x + \alpha_2.$$

$$\Rightarrow \begin{cases} \alpha_3 = 0 \\ \alpha_2 = 0 \\ 2\alpha_3 + \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \\ \alpha_3 = 0. \end{cases}$$

$\Rightarrow \{x, x^2+1, x^3+2x\}$ is linearly independent.

$\Rightarrow \{x, x^2+1, x^3+2x\}$ is a basis for $R(T)$.

for $N(T)$ $T(f(x)) = 0$

$\Rightarrow xf(x) + f'(x) = 0.$

suppose that $f(x) = a_2x^2 + a_1x + a_0$

$\Rightarrow a_2x^3 + a_1x^2 + a_0x + 2a_2x + a_1 = 0.$

$\Rightarrow a_2x^3 + a_1x^2 + (a_0 + 2a_2)x + a_1 = 0.$

$$\Rightarrow \begin{cases} a_2 = 0 \\ a_1 = 0 \\ a_0 + 2a_2 = 0 \\ a_1 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_0 = 0 \\ a_2 = 0 \end{cases}$$

$\Rightarrow f(x) = 0$ $\Rightarrow N(T) = \{0\}$, the basis of $N(T)$ is ϕ

③ $\text{rank}(T) = \dim(R(T)) = 3$

$\text{nullity}(T) = \dim(N(T)) = 0.$

④ $\text{rank}(T) = 3$ $\text{nullity}(T) = 0.$

$\dim P_2(\mathbb{R}) = 3$ $\Rightarrow \text{rank}(T) + \text{nullity}(T) = \dim P_2(\mathbb{R})$

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17. proof:

(a) $\because \dim V < \dim W$

$\therefore nullity(T) + rank(T) < \dim W$

$\because nullity(T) \geq 0$

$\therefore rank(T) < \dim W$

$\therefore \dim(R(T)) < \dim W$

and $\because R(T) \subseteq W$ by Thm. 1.11

$\therefore R(T) \subsetneq W$.

$\therefore$ T cannot be onto.

(b) $\because \dim V > \dim W$

$\therefore nullity(T) + rank(T) > \dim W$.

$\therefore nullity(T) > \dim W - rank(T)$
$= \dim W - \dim(R(T)) \geq 0$.

$\therefore nullity(T) > 0$.

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$\because T(0) = 0$.

$\therefore 0 \in N(T)$

$\because \dim\{0\} = 0$

$\therefore N(T)$ contains elements other than 0.

i.e. there are at least 2 vectors in V will be transferred to 0_W in W by W.

Hence T cannot be one-to-one.

Hence