Problem Set 8 — MAT257

November 23, 2016

Problems marked with $*$ are to be sumbitted for credit.

1 Munkres §12 (p.103)

1. Carry out Step 2 of the proof of Theorem 12.2.

2. Let $I = [0,1]$; let $Q = I \times I$. Define $f : Q \to \mathbb{R}$ by letting $f(x,y) = 1/q$ if y is rational and $x = p/q$ where p and q are positive integers with no common factor; let $f(x,y) = 0$ otherwise.

   (a) Show that $\int_Q f$ exists.

   (b) Compute

   $$\underline{\int_{y\in I}} f(x,y) \quad \text{and} \quad \overline{\int_{y\in I}} f(x,y).$$

   (c) Verify Fubini's theorem.

$*$ 3. Let $Q = A \times B$ where A is a rectangle in $\mathbb{R}^k$ and B is a rectangle in $\mathbb{R}^n$. Let $f : Q \to \mathbb{R}$ be a bounded function.

   (a) Let g be a function such that

   $$\underline{\int_{\mathbf{y}\in B}} f(\mathbf{x},\mathbf{y}) \le g(\mathbf{x}) \le \overline{\int_{\mathbf{y}\in B}} f(\mathbf{x},\mathbf{y})$$

   for all $\mathbf{x} \in A$. Show that if f is integrable over Q, then g is integrable over A, and $\int_Q f = \int_A g$.

   (b) Give an example where $\int_Q f$ exists and one of the iterated integrals

   $$\int_{\mathbf{x}\in A}\int_{\mathbf{y}\in B} f(\mathbf{x},\mathbf{y}) \quad \text{and} \quad \int_{\mathbf{y}\in B}\int_{\mathbf{x}\in A} f(\mathbf{x},\mathbf{y})$$

   exists, but the other does not.

   (c) Find an example where both the iterated integrals of (b) exist, but the integral $\int_Q f$ does not.

4. Let A be open in $\mathbb{R}^2$; let $f : A \to \mathbb{R}$ be of class $\mathcal{C}^2$. Let Q be a rectangle contained in A.

   (a) Use Fubini's theorem and the fundamental theorem of calculus to show that

   $$\int_Q D_2D_1f = \int_Q D_1D_2f.$$

   (b) Give a proof, independent of the one given in §6, that $D_2D_1f(\mathbf{x}) = D_1D_2f(\mathbf{x})$ for each $\mathbf{x} \in A$.

2 "In addition. . . "

$*$ A. Let $Q = [0,1]^3$ and let $f : Q \to \mathbb{R}$ be given by

$$f(x,y,z) = \begin{cases} 1, & x < y < z \\ 0, & \text{otherwise} \end{cases} .$$

Compute $\int_Q f$. (You may assume without proof that f is integrable.)