

## Notes on Orientations.

\* Let  $V$  be an  $k$ -dimensional vector space.

If  $v, u$  are ordered bases of  $V$ ,  $v \sim u$  if  $\det C_v^u > 0$  where  $C_v^u$  is the change of basis matrix

• Thm:  $\sim$  splits bases of  $V$  into two equivalence classes

pf: Let  $u, v, z$  be bases of  $V$ .

Reflexivity:  $C_u^u = I_{k \times k}$ , so,  $\det C_u^u > 0$ , so,  $u \sim u$

Symmetry:  $u \sim v \Rightarrow \det C_u^v > 0$ .  $\det C_v^u = \det(C_u^v)^{-1} = [\det(C_u^v)]^{-1} > 0$ , so,  $v \sim u$

Transitivity:  $v \sim u, u \sim z \Rightarrow \det C_v^u, \det C_u^z > 0 \Rightarrow \det C_v^z = \det[C^z_u C_u^z] = \det C_u^z \det C_v^u > 0$ , so,  $v \sim z$

Split into two: Assume  $u \not\sim v, z \Rightarrow \det C_v^u, \det C_u^z < 0 \Rightarrow \det C_v^z = \det[C^z_u C_u^z] = \det C_u^z \det C_v^u < 0$ , so,  $v \not\sim z$

• An orientation of  $V$  is an equivalence class of bases.

\* Lemma: Let  $W$  be an  $n$ -dimensional vector space,  $\phi: V \rightarrow W$  be linear & injective,  $u, v$  bases of  $V$

$u \sim v$  iff.  $\phi(u) \sim \phi(v)$  as bases of  $\text{Im } \phi$

pf: Let  $u = \{u_i\}_{i=1}^k, v = \{v_i\}_{i=1}^k \mid \forall i \in \mathbb{N}, u_i = \sum_j a_{ij} v_j, \text{ so, } \phi(u) = \sum_j a_{ij} \phi(v_j)$

$\phi$  is injective so  $\{\phi(u_i)\}_{i=1}^k, \{\phi(v_i)\}_{i=1}^k$  are bases of  $\text{Im } \phi$ .  $(\frac{\partial(\phi(v))}{\partial(u)} = \frac{\partial(\phi(v_1), \dots, \phi(v_k))}{\partial(u_1, \dots, u_k)} = C_v^u)$  so,  $u \sim v$  iff.  $\phi(u) \sim \phi(v)$

What follows is a reconciliation of what Dror has said & what the textbook has said about orientations. The important conclusions are in the second paragraph of page 5 & the 2<sup>nd</sup> paragraph of page 6, with the rest being proofs of concepts in class given without proof. Everything\* Dror & the text have mentioned about orientation are in these notes.

I've probably made mistakes; let me know if you find any!

\* The one thing left out & when Dror & the book seem to disagree is discussion on 1-manifolds & allowing patches to have domain  $L^2 \equiv \{x \in \mathbb{R} \mid x \leq 0\}$ . Dror didn't mention this in class as his definition of orientability didn't need it.