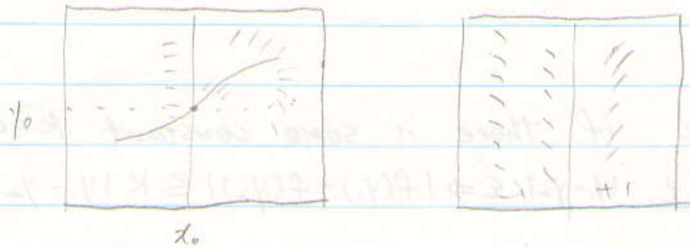
Sep 25, 2012 Mat267

Desired Theorem

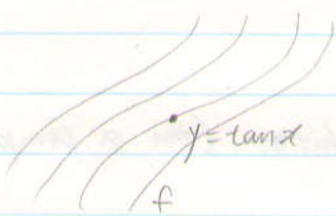
Given $f(x,y)$, $\phi' = f(x,\phi)$ with $\phi(x_0) = y_0$ has a solution and it is unique.

Problem 1

1. We hope unless f is at least continuous.



2. Even if f is super-continuous, the solution may exist only for a finite amount of time.



3. A solution may not be unique.

$$\sqrt[3]{y} = x - c$$

$$\underbrace{\sqrt[3]{y} - x}_{\psi} = c$$

$$\psi_x + \psi_y y' = 0$$

$$-1 + \frac{1}{3} y^{-\frac{2}{3}} y' = 0$$

$$y^{-\frac{2}{3}} y' = 3$$

$$\boxed{y' = \underbrace{3y^{\frac{2}{3}}}_{f(x,y)}}$$

Exercise

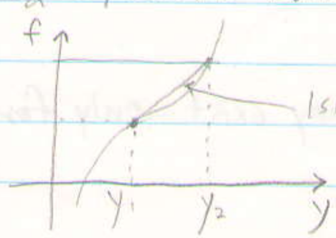
For any constant C,

$$\phi_c(x) = \begin{cases} 0 & x \le c \\ (x-c)^3 & x \ge c \end{cases}$$

is differentiable, solution equation & for all $c>0$, $\phi_c(0)=0$

Definition

$f: \mathbb{R}_y \to \mathbb{R}$ is called Lipschitz if there is some constant $k>0$, $\epsilon>0$ "the a Lipschitz constant of f" s.t. $|y_1-y_2|<\epsilon \Rightarrow |f(y_1)-f(y_2)| \le k|y_1-y_2|$



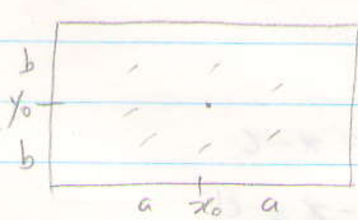
1. Lipschitz $\Rightarrow$ uniform continuous
2. f' exists & is continuous $\Rightarrow$ f is Lipschitz. (on a compact set)

Theorem (The Fundamental Theorem of ODEs or Picard's Theorem or ~~Existence~~ & Uniqueness Theorem for ODEs)

Let $f: R = [x_0-a, x_0+a] \times [y_0-b, y_0+b] \to \mathbb{R}$ be is continuous & uniformly Lipschitz relative to y.

$\exists K>0$, s.t. $\forall x, y_1, y_2$, $|f(x,y_1)-f(x,y_2)| < K|y_1-y_2|$

then the equation $\phi' = f(x,\phi)$ with $\phi(x_0)=y_0$ has a unique solution $\phi: [x_0-\delta, x_0+\delta] \to \mathbb{R}$



where $\delta = \min(a, \frac{b}{M})$ where M is a bound on f on R

proof

Equation $\Longleftrightarrow$ $\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt$

$\phi_0, \phi_1, \phi_2, \cdots \longrightarrow \phi$

First $\phi_0(x) = y_0$, then $\phi_n(x) = y_0 + \int_{x_0}^{x} f(t), \phi_{n-1}(t))\,dt$

Claim

1. ϕ_n is well-defined
2. For $n \geq 1$, $|\phi_n(x) - \phi_{n-1}(x)| < \frac{MK^{n-1}|x-x_0|^n}{n!}$