

MATH 240

Nov 7, 2016.

The bad news about Matrix Algebra:

1. $A \cdot B$ is defined only when dims match.
2. A^{-1} may not exist even if $A \neq 0$
- 3 In general, $A \cdot B \neq B \cdot A$.

Today's goals:

1. The good news about matrix algebra
2. A computational interlude.

~~Matrices have some Field properties:~~

1. $+, \times, 0, I_{n \times n}$
2. $A+B=B+A; A+(B+C)=(A+B)+C \dots \dots$ (works for matrices).
3. Associative laws: $A(BC)=(AB)C$

$$\underset{m \times n}{A} \times \underset{n \times n}{I} = A$$

$$\underset{m \times m}{I} \times \underset{m \times n}{A} = A$$

~~MAT~~

If $\exists B$ st. $B \cdot A = I$ and $A, B \in M_{n \times n}$, then $A \cdot B = I$.
 However we may have $A \in M_{m \times n}$, $B \in M_{n \times m}$ $A \cdot B = I_{m \times m}$
 Yet $B \cdot A \neq I_{n \times n}$.

4. $(A+B)C = AC + BC$
 $A(B+C) = AB + AC$

Proofs. Write all formulas and check that everything works.

Easier approach: All of these properties are very easy for linear transformations. Obviously $(T \circ S) \circ R = T \circ (S \circ R)$

$$\begin{array}{c} R \\ \xrightarrow{\quad C \quad} \\ S \\ \xrightarrow{\quad B \quad} \\ T \\ \xrightarrow{\quad A \quad} \end{array} \quad (AB) \cdot C = A(BC)$$

$$A \cdot I = A$$

$$\begin{bmatrix} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & 0 \end{bmatrix}$$

$$I \cdot A = A$$

"

$$\begin{bmatrix} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & 0 \end{bmatrix}$$

of 4:

$$\begin{array}{ccc} \checkmark & \xrightarrow{T} & \checkmark \\ \text{n-dim } S & \xleftarrow{\quad B \quad} & \text{n-dim } W \\ \mathcal{G} & & \mathcal{Y} \end{array}$$

$S \cdot A = I \Rightarrow S \circ T = I_{\mathcal{V}} \Rightarrow S$ is onto $\text{rank } S + \text{nullity } S = \dim W$
 $\Rightarrow \text{nullity } S = 0 \qquad n \qquad 0 \qquad n$
 $\Rightarrow S$ is 1-1
 $\Rightarrow S$ is invertible.

\Rightarrow true for general functions

$$\Rightarrow T \circ S = I$$

• $T: V \rightarrow W$

$$= (v_j)_{j=1}^n \quad \mathcal{V} = (w_k)_{k=1}^m$$

$$v \in V, w \in W \quad w = Tv$$

$$[v]_{\beta} = \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right]_n = v \quad [w]_{\gamma} = \left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right]_m = w \quad [T]_{\beta}^{\gamma} = A$$

$M_{m \times n}$

Claim: $\bar{w} = A \cdot \bar{v}$

$$\text{in } M_{m \times 1} \quad \text{in } M_{m \times n}$$

ordinary matrix multiplication

Pf. $A = (a_{kj}) \quad \bar{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow v = \sum b_j v_j \quad \bar{w} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \Rightarrow w = \sum c_k w_k$

$$\sum c_k w_k = w = Tv \quad T(\sum b_j v_j) = \sum_{j=1}^n b_j T(v_j) = \sum_{j=1}^n b_j \sum_{k=1}^m a_{kj} w_k = \sum_{k=1}^m \left(\sum_{j=1}^n a_{kj} b_j \right) w_k.$$

The w 's are a basis so the coeffs on both sides must be equal.

$$\Rightarrow c_k = \sum_{j=1}^n a_{kj} b_j. \quad \text{This is } \bar{w} = A \bar{v}.$$

(3)

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Compute the rank of.

$$\begin{bmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{bmatrix} \xrightarrow{\text{add } (-4) \text{ times row(1) to row(2)}} \begin{bmatrix} 4 & 4 & 4 & 8 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{bmatrix} \xrightarrow{\text{add } (-8) \text{ times row(1) to row(3)}} \begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -3 & -4 & -3 & 1 \\ 0 & -3 & -4 & -3 & 1 \end{bmatrix} \xrightarrow{\text{add } (-6) \text{ times row(1) to row(4)}} \begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -3 & -4 & -3 & 1 \\ 0 & -3 & -4 & -3 & 1 \end{bmatrix}$$

add(-4) times row(1) to row(2)

Technique (Thm)

The rank of a matrix does not change under row/column operations.

1. Interchange two rows/columns
2. Multiply a whole row/column by a scalar.
3. Add a multiple of one row/column to another

Add (-6) times row(1) to row(4).

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -3 & -4 & -3 & 1 \\ 0 & -3 & -4 & -3 & 1 \end{bmatrix} \xrightarrow{\text{add } (-6) \text{ times row(1) to row(4)}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -3 & -4 & -3 & 1 \\ 0 & -3 & -4 & -3 & 1 \end{bmatrix} \xrightarrow{\text{add } (-3) \text{ times row(1) to row(3)}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 2 & 0 & 4 \end{bmatrix} \xrightarrow{\text{add } (-2) \text{ times row(1) to row(4)}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 & 4 \end{bmatrix}$$

$\xrightarrow{\text{add } (-1) \text{ times row(1) to row(2)}}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 & 4 \end{bmatrix} \xrightarrow{\text{add } (-1) \text{ times row(1) to row(3)}}$$

$$\xrightarrow{\text{add } (-1) \text{ times row(1) to row(4)}}$$

$$\xrightarrow{\text{add } (-1) \text{ times row(2) to row(3)}}$$

$$\xrightarrow{\text{add } (-1) \text{ times row(2) to row(4)}}$$

$$\xrightarrow{\text{add } (-1) \text{ times row(3) to row(4)}}$$

$\underbrace{\quad\quad\quad}_{B}$

$$Tv_1 = w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Im } T = \text{span}(w_1, w_2, w_3)$$

(Range)

$$Tv_2 = w_2$$

$$Tv_3 = w_3$$

$$Tv_4 = \mathbf{0}$$

$$Tv_5 = \mathbf{0}$$

 $\therefore \text{rank } = 3 \text{ b/c } w_1, w_2, w_3 \text{ are lin. indep.}$

$$\xrightarrow{\text{V}} \xrightarrow{\text{I}} \text{W}$$

5-dim 4-dim

v_1, v_2, v_3, v_4 w_1, w_2, w_3