MAT327/December-06-10

Goal "Most" continuous functions $f: I \to \mathbb{R}$ are nowhere differentiable.

A "thin set" is closed & has empty interior.

Def A Baire space is a space X s.t. any countable collection of closed sets with no interior ("not chunky") in X has a union that has no interior.

Thm A complete metric space is Baire

Proof Let $F_1, F_2, \ldots$ be countably many closed sets having no interior.

Given any open $U \subset X$, want to show $U - \bigcup F_i \neq \emptyset$ (U not part of interior of $\bigcup F_i$).

Use induction: $U - F_1$ open (since F_1 closed) & non-empty (since $U \not\subset F_1$) so pick $x_1 \in U - F_1$ & $0 < r_1 < 1$ s.t. $\overline{B_{r_1}(x_1)} \subset U - F_1$

Now $B_{r_1}(x_1) - F_2$ is open & non-empty so pick $x_2 \in B_{r_1}(x_1) - F_2$ & $0 < r_2 < \frac{1}{2}$ s.t. $\overline{B_{r_2}(x_2)} \subset B_{r_1}(x_1) - F_2$

⋮ by induction, find sequence x_n, r_n s.t.

1. $0 < r_n < \frac{1}{n}$
2. $\overline{B_{r_n}(x_n)} \subset B_{r_{n-1}}(x_{n-1}) - F_n \subset B_{r_{n-2}}(x_{n-2}) - F_{n-1} - F_n \subset \cdots \subset U - \bigcup F_i$

Claim x_n is Cauchy: if $n, m > N, x_n, x_m \in B_{\frac{1}{N}}(x_N)$

Let $x = \lim_{n\to\infty} x_n$ (use compactness). Then $x \in \overline{B_{r_n}(x_n)} \subset B_{r_{n-1}}(x_{n-1}) - F_n$, so $x \notin F_n$ for any n, so $x \notin \bigcup F_n$, yet $x \in B_{r_1}(x_1) \subset U$ so $U - \bigcup F_i \neq \emptyset$ as required■

Thm (variant) A compact T_2 space is also Baire. Pf sketch: nest closures s.t. their intersection is non-empty using the FIP.

Restate A space is Baire iff every countable collection of open dense (complement of thin) sets has a dense intersection. In a Baire space, a countable intersection of open dense sets is "big".

Claim $X = C([0,1], d_\infty(f,g) = \sup|f(x) - g(x)|)$ is complete & hence Baire. In it, the set of cont. nowhere differentiable fns is "big".

X is complete: if f_n is a Cauchy sequence of fns then for any $x \in I$, the sequence $f_n(x)$ is itself Cauchy so it has a limit, $f(x)$. Exercise: show $f_n \to f$ uniformly, hence f is cont., hence $f \in X$, hence X is complete.

Left to show set of nowhere differentiable fns is a union of open dense sets. Let

$$U_n = \left\{ f: \begin{array}{c} \exists \text{ partition } 0 = x_0 < x_1 < \cdots < x_p = 1 \text{ s.t.} \\ 1. |x_{i+1} - x_i| < \dfrac{1}{n} \\ 2. \left|\dfrac{f(x_{i+1}) - f(x_i)}{|x_{i+1} - x_i|}\right| > n \end{array} \right\}$$

Then (1) U_n is open & dense and (2) If $f \in \bigcap U_n$ then f is cont. & nowhere differentiable.

Proof of 1

$\underline{U_n \text{ dense:}}$ define g that wiggles between f and $f-\varepsilon$, letting $x_i's$ be tips of wiggles, making slopes as big as we want.

$\underline{U_n \text{ open:}}$

$$U_n = \bigcup_{\substack{\text{all partitions} \\ 0=x_0<x_1<\cdots<x_p=1 \\ \text{s.t.} |x_{i+1}-x_i|<\frac{1}{n}}} \left\{ f : \Phi(f) = \left| \frac{f(x_{i+1}) - f(x_i)}{|x_{i+1} - x_i|} \right| > n \right\}$$

$$\Phi: X \to \mathbb{R} \text{ is cont.}$$

So U_n is open as the union of open sets under cont. fn:

$$U_n = \bigcup_{\text{all partitions}} \Phi^{-1}\big((n, \infty)\big) \blacksquare$$

<u>Proof of 2</u>

Suppose not. Use "triangle inequality for slopes" to bound slope in nbd of pt of differentiability. But $f \in \bigcap U_n$ means $f \in U_{1001}$ so can find partition that breaks bound, has slope $> 1001 >$ bound↯